GRADATIONS IN EUCLID:

BOOKS I. AND II.

AN INTRODUCTION TO PLANE GEOMETRY, ITS USE AND APPLICATION;

WITH

AN EXPLANATORY PREFACE,

REMARKS ON GEOMETRICAL REASONING, AND ON ARITHMETIC AND ALGEBRA APPLIED TO GEOMETRY;

PRACTICAL RESULTS AND EXERCISES:

BY

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"GEOMETRY IS, PERHAPS, OF ALL THE PARTS OF MATHEMATICS, THAT WHICH OUGHT TO BE TAUGHT FIRST: IT APPEARS TO ME VERY PROPER TO INTEREST CHILDREN, PROVIDED IT IS PRESENTED TO THEM PRINCIPALLY IN RELATION TO ITS APPLICATIONS, WHETHER ON PAPER, OR ON LAND."

—*Lacroix, p.* 306.

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PREFACE,

RESPECTING THE GRADATIONS AND SKELETON PROPOSITIONS, ETC.

THE chief aim of the Author or Compiler of the Gradations in Euclid, with Skeleton Propositions, &c., for Written Examinations, has been to furnish a *useful book* to those in the humbler stations of life, who attend our Parochial and similar Schools, who have not much time for the pursuit of Geometrical Studies, and to whom, therefore, the Practical Application of whatever they learn is of great importance. He is, however, persuaded that those who have both time and full opportunity, either in Public Schools or in Colleges, for attaining proficiency in the Higher Mathematics, will find an Introduction, such as is given in this work, very suitable to prepare them more thoroughly to appreciate Geometrical Truths, and to take an interest in them as the ground-work of accurate science.

The INTRODUCTION is of general use to all Students of Geometry: it contains—A brief account of the Gradual Growth of Geometry and of the Elements of Euclid;—The Signs and Contractions that may be employed: and some Remarks—on the Nature of Geometrical Reasoning,—on the Application of Arithmetic and Algebra to Geometry,—on Incommensurable Quantities,—and on Written and Oral Examinations. The subjects treated of pre-suppose, indeed, that the Learner has a clear understanding of Fractions, common and decimal,—of the extraction of the Square Root,—and of the introductory principles of Algebra: but this knowledge is indispensable for those who would really master the Elements of Geometry.

The Editions of *Euclid* by POTTS and BLAKELOCK, have shown the advantages of printing separately and distinctly the parts of a Proposition and of its Demonstration: it is a plan which undoubtedly gives very valuable help to Learners in attaining a more exact acquaintance with the Principles on which Geometry, as a science, is founded. No argument is here needed to prove the importance of being able to estimate the force and certainty of demonstrable truths:—it is the first condition of success, and the sure means of proficiency in Geometrical and in all other Studies.

The GRADATIONS IN EUCLID endeavour to carry out the Plan to a greater extent, and with increased distinctness. The propositions throughout are separated into successive steps; and in the margin, between the vertical lines, direct references are made to the reasons,—the definitions, axioms, or preceding propositions,—on which they depend. The method of printing, which has been adopted, also gives a clearer view both of the whole Proposition and of its parts; and familiarises the mind to an orderly and systematic arrangement,—so important an auxiliary to all sound progress. By following out a plan of this kind, Learners can scarcely fail to form a distinct conception, of what they have to do or to prove, and of the means by which their purpose is to be accomplished.

The EXPLANATORY NOTES direct the Learner's attention to several points of interest connected with the Definitions and Propositions; and to many of the Propositions is appended an account of the PRACTICAL USES to which the proposition may be applied. This is valuable for many reasons,—but chiefly that the Learner may at once see, not simply the theoretical and abstract truths of Geometry, but their direct utility in various ways. There are very many persons who, from studying only the Common Editions of Euclid, which treat exclusively of the Theory of Geometry, never attain to a perception of its importance, and never realize the full advantages of geometrical studies. It is the *main object* of the Gradations in Euclid *to combine Theory and Practice;* and as soon as a geometrical truth has been established, *to point out its use and application.* The Author is thoroughly persuaded that this *immediate* Combination of Theory and its Application, not only awakens and maintains a livelier interest, but in fact leads to a more scholar-like understanding of both, than when they are studied separately, or at wide intervals of time.

Only a *portion* of the Uses to which the Propositions may be applied has been given,—more in the way of example, and to point out in some instances the progress of geometrical discovery, than with the view of exhausting the subject. The various works on Practical Mathematics will supply what may be wanting in this respect. For the developement of the Uses and Applications of the First and Second Books of Euclid, geometrical principles not worked out in those books must occasionally be introduced; and though it is not strictly logical to employ truths that have not already been established, as the ground-work of further reasoning,—now and then, in this part of the work, *Lemmas*, or truths borrowed from another part of the subject, will be adopted as the foundation of new truths.

The PRACTICAL RESULTS will, it is presumed, be instructive to the Learner in various ways;—but especially as exhibiting a synoptical view of all the Problems contained in the Elements of Plane Geometry. He will not, indeed, have arrived at the means of demonstrating the problems, which lie out of the First and Second Books; but, inasmuch as their construction depends almost entirely on those two books, he will possess, for *practical purposes*, a knowledge of the methods by which Geometrical Figures, not being sections of a Cone, are to be constructed.

The APPENDIX, containing GEOMETRICAL ANALYSIS and GEOMETRICAL EXERCISES, appeared to the Writer needed for the completion of his plan, *i. e.*, for comprising a systematic teaching of Geometry, as far as the first and second books furnish the means of doing it. The Appendix and a Key to the Exercises, will each be published separately from the Gradations.

The SKELETON PROPOSITIONS, &c., *for pen-and-ink examinations*, are arranged and will be published in two Series,—one with the references in the margin; the other without those references. The *first* Series is intended for beginners; the *second*, for those who may be reasonably supposed to be prepared for a strict examination. The two Series will be found well adapted to test the Progress of the Learner, and to ascertain how far his knowledge of geometrical principles, and his power to apply them, really extend. The object is, in the *first* Series, to furnish the Learner, step by step, with the truths from which other truths are to be evolved, but to leave him to work out the results, and from the results, as they arise, to aim at more advanced conclusions: in the *second* Series,

where there are no references in the margin, the object is to make the examinations strict and thorough, yet so as to be conducted on one uniform plan. This uniformity will be found greatly to assist Examiners, when they compare the examination papers together for the purpose of deciding on their respective merits.

The Skeleton Propositions may be used either simultaneously with the Gradations, or, after the first and second books have been read in any of the usual editions of SIMSON'S *Euclid*, as a recapitulation of the ground already gone over: if used *simultaneously*, the Learner must first study the Definitions and Propositions in their order, and then, laying the Gradations aside, reduce his knowledge to a written form, as the references indicate in the vertical columns of the Skeleton Propositions; but if used as a recapitulatory exercise, a course in some respects different is recommended.

In the *recapitulatory exercise*, the following plan is recommended for adoption:—*first*, that the Learner should give in writing a statement of the meaning of various Geometrical Terms, of the nature of Geometrical Reasoning, and of the application of Algebra and Arithmetic to Geometry; *secondly*, that he should fill in,—*not by copying from any book, but from the stores of his own mind and thought*, trained by previous study of the GRADATIONS or of some similar work,—the Definitions, Postulates, and Axioms of which the leading words are printed; and *thirdly*, that he should proceed to take the Propositions in order, and write out the proofs at large, as the printed forms and references in the margin indicate: this should be done systematically in all the propositions, beginning with those truths already established which are required for the Construction and Demonstration, and then taking in order the Exposition, the Data and Quæsita, or the Hypothesis and Conclusion, the Construction with its methods, and the Demonstration with its proofs separated from each other, and given, step by step, in regular progression.

For the *thorough Examinations*, that series of the Skeleton Propositions must be used which contains the General Enunciation only, without any references printed in the margin. The Spaces for the exposition, construction, and demonstration, are retained, and also the vertical lines within which the Learners themselves are to place the references; but this is done simply for the purpose of securing a uniformity of plan in the

written examinations, and for the convenience of Examiners. The Student under examination should be required to write out the propositions, &c.,* needed in the Construction and Demonstration, and to supply the references to the various geometrical truths by which the steps of the proposition are established.

No figures, or diagrams, are given in either of the Series of Skeleton Propositions; as it is more conducive to the Learner's sound progress that he be left entirely to himself to construct these.

The Uses and Applications of the Propositions, at least *in a brief way*, —and where requisite, the Algebraical and Arithmetical Illustrations,— should not be neglected: it is in these that the practical advantage of abstract truths is rendered apparent.

It is imperative that the Teacher should revise each Proposition after it has been written out, and note all misapprehensions and inaccuracies before the Learner proceeds to the following proposition. In Self-Tuition the Learner must consult the Gradations, and by them correct the already filled-up Skeleton; but he must be faithful to himself, and to his own improvement, by not consulting the Gradations *as a Key*, until he has first worked out and written down his own conception of what the demonstration demands. He will thus build up *for* himself and *of* himself; he will make the dead bones of the Skeleton live, clothe them with flesh and sinews, and round them off in all their proper proportions.

A course of this kind followed faithfully through two books of the Elements of Geometry, will scarcely fail to render the Student competent by himself to master the other books of Euclid; and, should he desire it, by the same means. He will have learned the value of method and exactness; and expert in these, he will attain a solid and durable knowledge of Geometrical Principles.

At the present day nearly every edition of Euclid's Elements must be, more or less, a compilation, in which the Author draws freely on the labours of his predecessors. "The Gradations" are, in a great degree, of this character; and an open acknowledgment will suffice, once for all, to

* This is required on the principle that the repetition of old truths gains for them a more permanent residence in the mind.

repel any charge of intentionally claiming what belongs to others. It is affectation to pretend to great originality on a subject which has, like Geometry, for so many centuries exercised men's minds. If by the methods employed in the following pages, the Study of Geometry be rendered more interesting and more practically useful,—and especially if his work be adapted to the wants of that numerous class of Learners, the Pupils in Parochial and similar Schools,—the objects of the Editor will be accomplished. He desires no worthier calling than to be a fellow-labourer with the many excellent and talented Masters to whom, in the National, British, and other Public Schools of the United Kingdom, the responsibility is entrusted of training the young in sound learning.

The principal editions of Euclid to which the Editor is under obligation, are those of POTTS and LARDNER, and of an old Writer who professes to give "the uses of each Proposition in all the parts of the Mathematicks." An Exemplification and recommendation of the plan pursued in the Gradations and in the Skeleton Propositions, may be found in the preface to LARDNER'S *Euclid*, and in a *Treatise on the Study and Difficulties of Mathematics*, p. 74, attributed to Professor DE MORGAN. He is also indebted to various persons, Schoolmasters and others, for valuable suggestions, which he takes this opportunity to acknowledge.

The work is longer than the Author at first contemplated; but he trusts that the additions,—especially the Practical Results, and the Exercises,—will add considerably to its usefulness and value.

INTRODUCTION.

SECTION I.

GRADUAL GROWTH OF GEOMETRY AND OF THE ELEMENTS OF EUCLID.

Geometry has been defined in general terms to be,—"the Science of Space." It investigates the properties of lines, surfaces, and solids, and the relations which exist between them. *Plane Geometry* investigates the properties of space under the two aspects of length and breadth; *Solid Geometry*, under the three,—of length, breadth, and thickness. It is the consideration of the Elements of Plane Geometry on which we are about to enter.

Geometry, *land-measuring*, as the word denotes (from *gè*, earth or land, and *metron*, a measure), was in its origin an Art, and not a Science: it embraced probably a system of rules, more or less complete, for performing the simpler operations of land-surveying; but these rules rested on no regularly demonstrated principles,—they were the offspring rather of experiment and individual skill, than of scientific research. In the same way poems—even some of the noblest—were composed before the principles of poetry had been collected into a system; languages were spoken, long before a grammar had been compiled; and men reasoned and debated before they possessed either a logic or a rhetoric: so measurements were made, while as yet there was no accurate theory of measuring,—no abstract speculations concerning space and its properties.

The points and lines of such a Geometry were necessarily visible quantities. A mark, which men could see, would be their point; a measuring rod, or string, which they could handle, their line; a wall, or a hedge, or a mound of earth, their boundary. The first advance beyond this towards an

abstract Geometry, would be to identify the instruments which they used in measuring, with the lines and boundaries themselves; the finger's breadth, or the cubit, the foot, or the pace, would become representatives of a certain length without reference to the shape. It was only as the ideas and perceptions of those who cultivated the art of measuring grew more refined and subtile, that a Geometry would be evolved, such as Mathematicians understand by the term, in which a point marks only position; a line, extension from point to point; and surface, a space enclosed by mathematical lines.

The Truths of Geometry, as a science, regularly as they are laid down and deduced in the Elements of Euclid, were not worked out by one mind, nor established in any systematic order. Some were discovered in one age, some, in another; two or three propositions by one philosopher, and two or three, by some one else. The collection of geometrical truths had thus a gradual growth, until it received completion at the hands of Euclid of Alexandria.

Thales, who predicted the eclipse that happened B. C. 609, is said to have brought Geometry from Egypt, and to have established by demonstration Propositions 5, 15, and 26, of Bk. i.; 31, iii.; and 2, 3, 4, and 5, of bk. iv. Pythagoras, born about 570 B. C., was the first who gave to Geometry a scientific form, and discovered Propositions 32 and 47 of bk. i.: Oenopides, a follower of Pythagoras, added the 12th and 23rd of bk. i.: and Eudoxas, B. C. 366, a friend of Plato, wrote the doctrine of proportion as developed in the fifth book of the Elements. These assertions may not rest on the firmest authority, yet they shew, even if they are only surmises, that Geometry was regarded by the Greeks as a science of very gradual formation, receiving accessions from age to age, and from various countries. It was at first a set of rules, until philosophy investigated the principles on which the rules were founded, and out of the chaos created knowledge.

According to Proclus, EUCLID of Alexandria flourished in the reign of the first Ptolemy, B. C. 323–283. To him belongs the glory, for such it is, of having collected into a well-arranged system, the scattered principles and truths of Geometry, and of having produced a work, which, after standing the test of above twenty centuries, seems destined to remain the Standard Geometry for ages to come.

Euclid's work comprises thirteen books, of which the first four and the sixth treat of Plane Geometry; the fifth, of the Theory of Proportion, applicable to magnitude in general; the seventh, eighth, and ninth, are on Arithmetic; the tenth, on the Arithmetical Characteristics of the divisions of a straight line; the eleventh and twelfth, on the Elements of Solids; and the thirteenth, on the Regular Solids. To the thirteen books by Euclid, Hypsicles of Alexandria, about A.D. 170, added the fourteenth and fifteenth books,—also on the Regular or Platonic Solids.

In modern times it is not usual to read more than six books of Euclid's Elements. The seventh, eighth, ninth, and tenth books treat of Arithmetic and of the Doctrine of Incommensurables, and have no proper connection with the first six books; and the eleventh and twelfth books, comprehending the First Principles of Solid Geometry, are to a considerable degree superseded by other Treatises.

Of the Six Books, the *first* may be described in general terms as treating of the Geometry of Plane Triangles; the *second*, of Rectangles upon the parts into which a straight line may be divided; the *third* book, of those Properties of the Circle which can be deduced from the preceding books; the *fourth* book, of such regular and straight-lined figures as can be described in or about a circle; the *fifth*, of Proportion with regard to magnitude in general; and the *sixth*, of similar figures, and of Proportion as applied to Geometry.

Our proposed limits confine us, for the present at least, to the first and second books. The first book, besides the Definitions, Postulates, and Axioms, contains forty-eight propositions, of which fourteen are problems for giving power to construct various lines, angles, and figures; and thirty-four are theorems, being the expositions of new geometrical truths. Of these theorems, some may be regarded as simply subsidiary to the proof of others more important, and of wider and more general application. The Propositions to be ranked among those of high importance, are Props. 4, 8, and 26, containing the criteria of the equality of triangles: Prop. 32 proving that the three interior angles of every triangle are together equal to two right angles: Prop. 41 declaring that a parallelogram on the same base and of the same altitude as a triangle, is double of the triangle: and Prop. 47 demonstrating that the square on the hypotenuse of a

right-angled triangle, is equal to the sum of the squares on the base and perpendicular.

The second book treats of the properties of Right-Angled Parallelograms, contained by the parts of divided straight lines. There are fourteen Propositions; of which Props. 11 and 14 are problems,—the other twelve are theorems. Props. 12 and 13 give the Elements of Trigonometrical Analysis, or the Arithmetic of sines, and are of great use in the Higher Geometry: the other propositions may be classified according to the mode of dividing the line or lines; Prop. 1 relating to the rectangles formed by one undivided line and the parts of a divided line; Props. 2, 3, 4, 7, and 8, to the rectangles formed by a line and any two parts into which it may be divided; Props. 5 and 9, to the rectangles on a line divided equally and unequally; and Props. 6 and 10, to the rectangles formed on a line bisected and produced.

The English Translation of Euclid, published by Dr. Robert Simson, of Glasgow, in 1756, has nearly, in some form or other, superseded all others, and is considered the standard text of an English Euclid. As containing "the Elements of Geometry," it is "unexceptionable, but is not calculated to give the scholar a proper idea of the Elements of Euclid," as Euclid himself left them. Various alterations, additions, and improvements, were made by Simson: but "with the exception of the editorial fancy about the perfect restoration of Euclid, there is little to object to in this celebrated edition. It might indeed have been expected that some notice would have been taken of various points on which Euclid has evidently fallen short of that formality of rigour which is tacitly claimed for him. We prefer," says De Morgan, "this edition very much to many which have been fashioned upon it,—particularly to those which have introduced algebraical symbols into the demonstrations in such a manner as to confuse geometrical demonstration with algebraical demonstration."—(See the Article, *Eucleides*, by De Morgan, in *Smith's Dictionary of Greek and Roman Biography*, Vol. II., pp. 63–74.)

In the face of such authority, it may seem bold to advocate the use of a Symbolical Notation; but, within certain limitations, the symbols of Arithmetic and of Algebra have a universal meaning, and may therefore be employed without any disadvantage, and certainly without confusion in our ideas. The pre-

caution needed, is, that we take care not to depart from the strictly geometrical application.

For an outline of the origin and progress of the science of Geometry, the learner should consult the *Introduction to the Elements of Euclid*, edited by Robert Potts, M.A., Trinity College, Cambridge.

SECTION II.

SYMBOLICAL NOTATION AND ABBREVIATIONS THAT *MAY* BE USED.

I.—*Signs common to Arithmetic, Algebra, and Geometry.*

$\because$ because.
$\therefore$ therefore.
$=$ equals, *or* equal.
$\neq$ not equal.
$>$ greater than.
$\ngtr$ not greater than.
$<$ less than.
$\nless$ not less than.
$+$ *plus*, sign of adding.
$-$ *minus*, sign of subtracting.
$\sim$ sign of difference between.

II.—*Geometrical Signs.*

· a point.
| straight line.
|| parallel to.
||s parallels.*
$\angle$ angle.
$\triangle$ triangle.
$\perp$ perpendicular to, or at right angles.
▱ parallelogram.
□ rectangle.
$\odot$ circle.
$\odot$ce circumference.

* When an *s* is added to the sign, the plural is denoted.

A single capital letter, as A, or B, denotes the point A, or the point B.

Two capital letters, as A B, or C D, denote the straight line A B, or C D.

Two capital letters, with the figure ^{2} just above to the right hand, as AB^2, denote, not the square of A B, but the square on the line A B.

Capital letters, with a point between them, as A B . C D, denote, not the product of A B multiplied by C D, but the rectangle formed by two of its sides meeting in a common point.

III.—*Abbreviations.*

Add. ....*Addendo*, by adding.
Ax. ......Axiom.
Conc. ....Conclusion, inference.
Cor. ......Corollary.
Cons. ....Construction.
C. 1 &c. ..Step 1 &c. of the Construction.
Dat. ......*Datum*, or *data*.
Def. ......Definition.
Dem. ....Demonstration.
D. 1 &c. ..Step 1 &c. of the Dem.
Exp. ....Exposition, or Particular enunciation.
Gen. ....General enunciation.
Hyp. ....Hypothesis.
H. 1 &c. ..Step 1 &c. of the Hyp.
Prob. ....Problem.
P. or Prs. Proposition or Propositions.
Pst. or Psts. Postulate or Postulates.
Quæs. ....*Quæsitum* or *Quæsita*.
Rec. ......Recapitulation.
Remk. ....Remark to be made.
Sch. ......*Scholium* or *Scholia*.
Sim. ....Similarly.
Sol. ......Solution.
Sub. ....*Subtrahendo*, by subtracting.
Sup. ......Suppose.
Super. ....*Superponendo*, by superposition.
Theor. ....Theorem.

Q.E.D., *quod erat demonstrandum*, which was the thing to be proved.
Q.E.F., *quod erat faciendum*, which was the thing to be done.

ad imposs. *ad impossibile*.
a fort. ....*a fortiori*, by a stronger argument or reason.
alt. ......altitude.
altr. ......alternate.
assum. ....*assumendo*, by adopting or taking.
bis. ......bisect.
bisd. ....bisected.
bisg. ....bisecting.
com. ....common.
con. sup...contrary supposition.
c. sc. ....circumscribe.
d. sc. ....describe.
eq. ang. ..equiangular.
eq. lat. ....equilateral.
ex. ab. ....*ex absurdo*, by an absurdity.
ext. ......exterior.
int. ......interior.
magn. ....magnitude.
opp. ......opposite.
par. ......parallel.
parlm. ....parallelogram.
pos. ......position.
qu. ang. ..quadrangular.
qu. lat. ..quadrilateral.
rect. ....rectangle.
rectl. ....rectilineal.
rectr. ....rectangular.
rem. ....remaining.
rt. ......right.
sq. ......square.
st. ......straight.

SECTION III.

EXPLANATION OF SOME GEOMETRICAL TERMS.

A *Definition* is a short description of a thing by such of its properties as serve to distinguish it from all other things of the same kind.

A *Postulate* is a self-evident problem, the admission of which is demanded without formal proof.

An *Axiom* is a self-evident theorem, or the assertion of a truth, which does not need demonstration;—it is worthy of credit as soon as stated.

A *Proposition* is something proposed to be done, as a problem; or to be proved, as a theorem.

A *Problem* is a proposal to do a thing, to construct a figure, or to solve a question.

A *Theorem* is the assertion of a geometrical truth, and requires demonstration.

The *Data* are the things granted in a problem;

The *Quæsita* are the things sought for in it;

The *Hypothesis* is the supposition made in a theorem;

The *Conclusion* is the consequence or inference deduced from it.

The *General Enunciation* of a proposition sets forth in general terms the conditions of the problem, or theorem, with what has to be done, or with what is inferred or concluded.

The *Exposition*, or *Particular Enunciation*, sets forth the same conditions with an especial reference to a figure that has been drawn.

The *Solution* of a problem shows how the thing proposed may be done.

The *Construction* prepares, by the drawing of lines, &c., for the demonstration of a proposition.

The *Demonstration* proves that the process indicated in the solution is sound, or that the conclusion deduced from an hypothesis, is true; *i. e.*, in accordance with geometrical principles.

The *Recapitulation*, or *Conclusion*, is simply the repetition of the proposition, or general enunciation, as a fact, or as a truth, with the declaration Q.E.F., or Q.E.D.

A *Corollary* is an inference made immediately from a proposition.

A *Scholium* is a note or explanatory observation.

A *Lemma* is a preparatory proposition borrowed from another part of the same subject, and introduced for the purpose of establishing a more important proposition.

The *Converse* of a proposition is when the hypothesis of a former proposition becomes the conclusion, or predicate, of the latter proposition.

The *Contrary* of a proposition is when that which the proposition assumes, is denied.

Direct Demonstration is when the very thing asserted is proved to be true.

Indirect Demonstration is when all other cases, or conditions, except the one in question, are proved not to be true, and the inference is made—therefore the very thing in question must be true; the assumption being that one out of several, or many, must be right.

The *Position* only of a line is meant, when the line is said to be given.

The *Length* only of a line is meant, when the line is said to be finite.

The *Base* of a figure is the side on which it appears to stand, but each side, in turn, with the position of the figure changed, may become the base.

The *Vertex* is the highest angular point of a figure: with a change of position in the figure, each angle may be named the vertical angle.

The *Subtend* of an angle is the side stretching across opposite to the angle.

The *Hypotenuse* is the subtend to a right angle.

The *Perpendicular* is the line forming with the base a right angle: lines are perpendicular to each other when at the point of junction they form a right angle.

A *Figure* is applied to a straight line when the line forms one of its boundaries.

The *Altitude* of a figure is the perpendicular distance from the side or angle opposite to the base, to the base itself, or to the base produced.

A *Diagonal* is a line joining two opposite angular points.

The *Complement* of an angle is what is wanted to make an acute angle equal to a right angle, or to 90°.

The *Supplement* of an angle is what is wanted to make an angle equal to two right angles, or to 180°.

The *Explement* of an angle is what is wanted to make an angle equal to four right angles, or to 360°.

The *Complements of a Parallelogram*, when the parallelogram is bisected by its diagonal, and subsidiary parallelograms are formed by two lines, one, parallel to one side, and the other, parallel to the other side, and both intersecting the diagonal,—the *complements of the parallelogram* are those subsidiary parallelograms through which the diagonal does not pass; and these, with the subsidiary parallelograms through which the diagonal does pass, fill up or complete the whole parallelogram.

The *Area of a Figure* is the quantity of surface contained in it, reckoned in square units, as square inches, square feet, &c.

A *locus* in Plane Geometry is a straight line, or a plane curve, every point of which, and none else, satisfies a certain condition.

SECTION IV.

NATURE OF GEOMETRICAL REASONING.

THE Demonstrations in Euclid's Elements of Geometry consist of arguments or reasonings by which the assertions made in the propositions are proved to be true. Thus, in the 15th Prop., bk. i., the assertion is made that "the opposite, or vertical angles, formed by two intersecting lines, are equal;" and the demonstration shows by argument, or reasoning, founded upon truths already admitted or proved, that the assertion itself must be received as true.

When fully stated, each argument contains both the thing which is proved, and the means by which the proof is established: and as in the arrangement of the parts of an argument the means of proof usually precede the thing proved, they are named the *premisses;* and the thing proved is named the *conclusion* or *inference*. Thus, in Prop. 1, bk. i., the premisses are—1st, things equal to the same thing are equal to each other;

2nd, the line A C, and also the line B C, are each equal to the same line A B; and 3rd, the inference, conclusion, or thing proved, is, that the line A C equals the line B C, or, adhering more strictly to the forms of reasoning, the two lines A C and B C are equal to each other.

Here in the premisses two things are laid down, or granted to be true: as,—"things equal to the same thing are equal to each other,"—this is *one truth;* "the line A C equals the line A B, and the line B C also equals the same line A B,"—this is *another truth;* and from the two things thus declared to be true, there is made in the conclusion the necessary and unavoidable inference,—therefore A C equals B C, *i. e.*, the two lines are equal to each other.

The subject of the conclusion, "the two lines A C and B C," is called the *minor* term; the predicate of the conclusion, "are equal to each other," is called the *major* term.

There are three terms in the premisses,—the major and the minor terms, and a third term with which as with a standard the major and the minor terms are compared. This third term, being the *medium* of the comparison, is named the *middle* term: it enters into the major premiss as the subject, and into the minor premiss as the predicate. In the example given, "things equal to the same," is the *middle* term, being the subject of the major premiss—"things equal to the same are equal to each other," and the predicate of the minor premiss—"the lines A C and B C are each equal to the same line A B."

The premiss in which the major term—*i. e.*, the predicate of the conclusion—appears, is called the *major* premiss; that in which the minor term, or subject of the conclusion, appears, is the *minor* premiss. Thus in the argument already given:

Major premiss, Because things equal to the same, are equal to each other;

Minor premiss, and because the two lines A C and B C, are each equal to the same line A B;

Conclusion, therefore the two lines A C and B C are equal to each other.

Here, "*the two lines A C, B C*" is the subject, and "*equal to each other*" the predicate of the conclusion;

"*The two lines A C, B C*" is the subject, and "*each equal to the same line A B,*" the predicate of the minor premiss;

"*Things equal to the same line A B*" the subject, and "*equal to each other*" the predicate of the major premiss.

Thus, "*equal to each other*" is the *major* term; "*the two lines A C, B C*" the *minor* term; and "*things equal to the same line A B*" the *middle* term of the argument.

In this mode of reasoning, it is seen that assertions are broadly made; and we may ask, on what *evidence* are these assertions themselves to be received as true?

The *first* kind of evidence is from the *definition* of the thing; thus, we define a triangle to be a figure bounded by three sides; and if, of any figure placed before us, we can affirm that it has three sides exactly, the conclusion is inevitable, that this figure also is a triangle.

The *second* kind of evidence is from the *axioms*, or truths so plain that they need no proof: for example,—we receive as undeniable, that, if equals be added to equals, the wholes are equal; and we argue, if to the line A D, or to its equal the line B C, we add another line E F, then the whole line made up of A D + E F, will equal the whole line made up of B C + E F.

The *third* kind of evidence is from the *hypothesis*, or supposition, which we make as the condition of our assertion: we declare, "in an isosceles triangle, the angles at the base are equal;" the very words, though not in the exact form of an hypothesis, directly imply the supposition, "if a triangle is isosceles," "then the angles at the base are equal." An isosceles triangle is here taken as the starting point of the reasoning;—and though, for the demonstration of the inference, "the angles at the base are equal," it is necessary to draw various lines which are not mentioned in the hypothesis, the conclusion at which we arrive is altogether dependent on the hypothesis.

The *fourth* kind of evidence is *from proof already given;* for what has once been established, may afterwards be taken for granted. For instance, when we have once established the truth, that "the interior angles of every triangle are together equal to two right angles," and we afterwards come to a proposition in the demonstration of which we need this established truth, we do not again go through all the steps by which the equality of the sum of the interior angles of a triangle to two right angles has been proved, but, without going down again to the bottom of the ladder before we make a step higher, we start

from the step we had already gained, and at once take up our position on a more advanced truth.

But the *Principle of Geometrical Reasoning* is, that from two propositions established or received as true, a third proposition, or inference, shall be made. Now, that this may be done, there must be something in common contained in both the propositions, with which common thing, the other two things are compared: we say—

All the triangle is in the circle,
All the square is in the triangle,
therefore, All the square is in the circle:

the common term of comparison here is "the triangle," and our inference is correct.

But if we say—

All the triangle is in the circle,
All the square is in the circle;
and infer, All the square is in the triangle;

this may be, or may not be,—for it may happen that only a part of the square is in the triangle. The fault of the apparent argument is, there is no proper term of comparison,—no middle term which is at the same time the subject of the major premiss, and the predicate of the minor premiss. To have the argument sound, we say—

Major P. All the triangle is in the circle,
Minor P. All the square is in the triangle,
therefore, All the square is in the circle.

When a connexion is thus declared to exist between the premisses and the conclusion,—that is, when reasons are stated and an inference made,—this mode of argument receives the name of a *Syllogism;* for a Syllogism is a bringing together into one view the two steps of the reasoning on which a truth depends, and the truth itself; or, as Whately, in his *Elements of Logic*, p. 52, defines a Syllogism, it is "an argument so expressed, that the conclusiveness of it is manifest from *the mere force of the expression*, *i.e.*, without considering the *meaning of the terms:* *e.g.*, in this Syllogism—

"Every Y is X,
" Z is Y,
therefore Z is X;

"the Conclusion is inevitable, whatever terms X, Y, and Z

respectively, are understood to stand for. And to this form all legitimate Arguments may ultimately be brought."

Every Syllogism is made up of three propositions, or assertions; of which, two are named the *premisses*, and the third, the *conclusion*. The Proposition which contains the *predicate* of the conclusion, is called the *major* premiss; and that which contains the *subject* of the conclusion, the *minor* premiss. The premisses are usually introduced by the word *because*, or by some similar word, and the conclusion by the word *therefore*.

The Syllogism is exhibited in *four forms*, or *figures*, distinguished from each other by the *situation* of the *Middle Term*, or Standard of Comparison, with respect to the *extremes* of the Conclusion,—that is, with respect to the major or minor terms. "The *proper order* is to place the Major premiss *first*, and the Minor, *second;* but this does not *constitute* the Major and Minor premisses; for that premiss (wherever placed) is the Major which *contains the major term*, and the Minor, *the minor*."—(*Elements of Logic*, p. 57.) The major term, as we have before said, is the predicate, and the minor term the subject of the conclusion.

Taking X to represent the Major term, Z the Minor term, and Y the Middle term, we may now exhibit the four forms of the Syllogism, of which four forms one or the other is used in all legitimate reasoning.

I. The *First* Form, or figure, of the Syllogism, which is also the clearest and most natural, makes the Middle term the *subject of the major premiss*, and the *predicate of the minor*. Thus, in Prop. 1, bk. i., of Euclid's Elements—

Major P.	Because $Y=X$,	or, because the line A B is equal to the line A C,
Minor P.	and $Z=Y$;	and the line B C is equal to the line A B;
Concl.	therefore $Z=X$.	therefore the line B C is equal to the line A C.

Here we may observe that B C, or Z, is the subject of the conclusion, and A C, or X, the predicate;
B C, or Z, the subject, and A B, or Y, the predicate of the minor premiss;
A B, or Y, the subject, and A C, or X, the predicate of the major premiss;
A C, or X, the major term; B C, or Z, the minor term; and A B, or Y, the middle term, or the standard of comparison.

II. In the *Second Form* of the Syllogism, the Middle term is the *predicate* of *both* premisses; as in Prop. 47, bk. i., of the Elements—

Major P.	Because $X = Y$,	or, because the angle DBC is a right angle,
Minor P.	and $Z = Y$;	and the angle FBA is a right angle;
Concl.	therefore $Z = X$.	therefore the angle FBA = the angle DBC.

Here ∠ F B A, or Z, is the subject, and ∠ D B C, or X, the predicate of the conclusion;
∠ F B A, or Z, is the subject, and rt. angle, or Y, the predicate of the minor premiss;
∠ D B C, or X, is the subject, and rt. angle, or Y, the predicate of the major premiss;
∠ D B C, or X, is the major term; ∠ F B A, or Z, the minor term; and *right angle*, the middle term.

III. In the *Third Form* of the Syllogism, the Middle term is the *subject* of *both* premisses; as in Prop. 28, bk. i., of the Elements—

Major P.	Because $Y = X$,	or, because ∠ EGB is equal to ∠ GHD,
Minor P.	and $Y = Z$;	and ∠ EGB is equal to ∠ AGH;
Concl.	therefore $Z = X$.	therefore ∠ AGH is equal to ∠ GHD.

Here Z, or ∠ A G H, is the subject, and X, or ∠ G H D, the predicate of the conclusion;
Y, or ∠ E G B, is the subject, and Z, or ∠ A G H, the predicate of the minor premiss;
Y, or ∠ E G B, is the subject, and X, or ∠ G H D, the predicate of the major premiss;
X, or ∠ G H D, represents the major term; Z, or ∠ A G H, the minor term; and Y, or ∠ E G B, the middle term.

IV. The *Fourth Form* of the Syllogism is the reverse of the first, and is the most unnatural of all: it places the Middle term the *predicate of the major premiss*, and the *subject of the minor*. Thus in Prop. 26, case 2, bk. i., of the Elements—

Major P.	Because $X = Y$,	or, because ∠ BHA is equal to ∠ EFD,
Minor P.	and $Y = Z$;	and ∠ EFD is equal to ∠ BCA;
Concl.	therefore $Z = X$.	therefore ∠ BCA is equal to ∠ BHA.

Here Z, or ∠ B C A, is the subject, and X, or ∠ B H A, the predicate of the conclusion;

Y, or $\angle$ E F D, is the subject, and Z, or $\angle$ B C A, the predicate of the minor premiss;

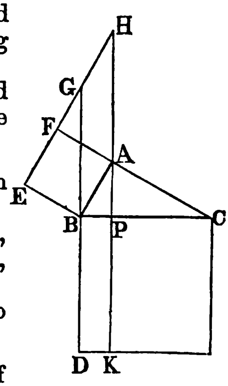
X, or $\angle$ B H A, is the subject, and Y, or $\angle$ E F D, the predicate of the major premiss;

and X, or $\angle$ B H A, is the major term; Z, or $\angle$ B C A, the minor; and Y, or $\angle$ E F D, the middle term.

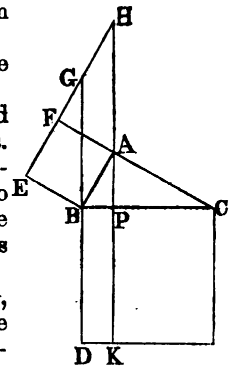
N.B.—This fourth form is never employed except by some accidental awkwardness of expression. Prop. 26, case 2, bk. i., would be much better arranged if we reduced it to the second form, and said—

Major P.	Because $X = Y$,	or, because	$\angle$ B H A equals $\angle$ E F D,
Minor P.	and $Z = Y$;	and	$\angle$ B C A equals $\angle$ E F D;
Concl.	therefore $Z = X$.	therefore	$\angle$ B C A equals $\angle$ B H A.

All legitimate arguments may be brought to one or the other of these four forms of the Syllogism; and in reading Euclid's Elements it will be an improving exercise, occasionally, to present the arguments in the regular Syllogistic form. We subjoin an example from *The Study and Difficulties of Mathematics.* pp. 73, 74: it is Prop. 47, bk. i., of the Elements. It will serve as a model for putting other propositions into the syllogistic form.



Exp.	1	Hyp.	Let A B C be a rt. angled triangle, B A C being the rt. angle;
	2	Concl.	the squares on A B and A C together equal the square on B C.
Cons.	1	46. I.	Describe the squares on B C and B A;
	2	Pst. 2.	produce D B to meet E F, produced if necessary, in G;
	3	31. I.	and draw H K parallel to B D and through A.
E M.	1	Def. 30.	I. Conterminous sides of a square are at rt. angles to one another;
		C. 1.	E B and B A are conterminous sides of a square;
			∴ E B and B A are at rt. angles.



DEM.	2	Sim.	II. By a similar Syllogism we prove that D B and B C are at rt. angles, and also that G B and B C are at rt. angles.
	3	D. 1 & 2.	III. Two right lines perpendicular to two other rt. lines make the same angle as those others:
		D. 1 & 2.	the lines E B and B G, A B and B C, are two rt. lines perpendicular, &c. ∴ the angle E B G is equal to the angle A B C.
	4	Def. 30. C. 1.	IV. All the sides of a square are equal: A B and B E are sides of a square; ∴ A B and B E are equal sides.
	5	Def. 10. C 1 & Hyp	V. All right angles are equal, $\angle$ B E G and $\angle$ B A C are rt. angles; ∴ $\angle$ B E G and $\angle$ B A C are equal angles.
	6	26. I.	VI. Two triangles having each two angles and the interjacent side equal, are equal in all respects:
		D. 3, 4, 5.	B E G and B A C are two triangles having the angles B E G and E B G respectively equal to $\angle$s B A C and A B C, and the side E B equal to the side B A; ∴ the triangles B E G and B A C are equal in all respects.
	7	D. 6. Def. 30.	VII. The side B G is equal to the side B C, and the side B C is equal to the side B D; ∴ the side B G is equal to the side B D.
	8	Def. A.	VIII. A four-sided figure of which the opposite sides are parallel, is a parallelogram:
		C. 1.	B G H A & B P K D are four-sided figures of which the opposite sides are parallel; ∴ B G H A & B P K D are parallelograms.
	9	35. I.	IX. Parallelograms upon the same base and between the same parallels, are equal:

DEM.	9	C. 3.	EBAF and BGHA are parallelograms on the same base, &c.; ∴ EBAF and BGHA are equal parallelograms.
	10	36. I.	x. Parallelograms on equal bases and between the same parallels, are equal:
		C. 3.	BGHA and BDKP are parallelograms on equal bases, &c.; ∴ BGHA and BDKP are equal.
	11	D. 9.	xi. The parallelogram EBAF is equal to parallelogram BGHA:
		D. 10.	parlm. BGHA is equal to parlm. BDKP; ∴ EBAF, the square on AB, is equal to parm. BDKP.
	12	Sim.	xii. A similar argument, from the commencement, proves that the square on AC is equal to the rectangle CPK.
	13	Ax. 8.	xiii. The rectangles BK and CK are together equal to the square on AB;
		D. 11 & 12	the squares on BA and AC are together equal to the rectangles BK and CK; ∴ the squares on BA and AC are together equal to the square on AB.

INDUCTION is a species of argument in which that is inferred respecting a whole class, which has been ascertained respecting several individuals of the class: carried out completely, *Inductive reasoning* is that in which a universal proposition is proved by proving separately each of its particular cases: thus, the angles A, B, C, and D are all the angles in a certain figure ABCD; we prove that A is a right angle, B a right angle, C a right angle, and D a right angle; and we say, therefore all the angles of the figure ABCD are right angles.

The argument '*a fortiori*,' by the stronger reason, proves that a given predicate belongs in a greater degree to one subject than to another: as, A is greater than B; B greater than C; much more, *a fortiori*, is A greater than C. An example of this kind of argument occurs in Prop. 21, bk. i., of the *Elements*: thus—

The angle BDC is greater than the angle CED;
and angle CED is greater than the angle BAC;
much more ∴ is angle BDC greater than angle BAC.

The '*reductio ad impossibile*,' the reduction to an impossibility, is when the argument shows that any given assertion is impossible; as in the 14th Prop., bk. i., where it is supposed that both the line B E and the line B D are continuations of another line C B; the demonstration conducts to the conclusion that the less angle A B E equals the greater angle A B D; but this is an impossibility, for the less cannot equal the greater.

There is also the '*reductio ad absurdum*,'' the reduction of an argument to an absurdity: it takes place when the conclusion, at which we arrive, involves something foolish, or utterly unreasonable: thus, in Prop. 7, bk. i., the angle B D C is proved, by the course of argument adopted, to be, first, equal to the angle B C D, and next, greater than the same angle B C D; but this is an absurdity. There is little real difference in Geometry between the impossible and the absurd.

"THE VALIDITY OF AN ARGUMENT depends upon two distinct considerations:—1, the truth of the relations assumed, or represented to have been proved before; 2, the manner in which these facts are combined so as to produce new relations:—in which last, the *reasoning* properly consists. If either of these be incorrect in any single point, the result is certainly false." "The same thing holds good in every species of reasoning; and it must be observed, however different geometrical argument may be in form from that which we employ daily, it is not different in reality." "The commonest actions of our lives are directed by processes exactly identical with those which enable us to pass from one proposition of geometry to another. A porter, for example, who being directed to carry a parcel from the City to a street which he has never heard of, and who, on enquiry, finding it is in the Borough, concludes that he must cross the water to get at it, has performed an act of reasoning, differing nothing in kind from those by a series of which, did he know the previous propositions, he might be convinced that the square of the hypotenuse of a right-angled triangle, is equal to the sum of the squares of the sides."—*On the Studies and Difficulties of Mathematics, p.* 76.

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SECTION V.

THE APPLICATION OF ALGEBRA AND ARITHMETIC TO GEOMETRY.

Algebraical Geometry, or the Application of Algebra to Geometry, has two provinces;—the one, when Algebra is employed for the investigation of geometrical theorems and problems, and as the great instrument of mathematical Analysis;—the other, when the Notation and Methods of operation usually appropriated to Algebra are introduced to give expression to geometrical truths, and to furnish the formulas according to which the Practical Geometrician must solve the questions which come before him. For success in the use of it, the first kind requires a more extended acquaintance both with Geometry and with Algebra than beginners in either science can have attained; the second kind, therefore, although it is not fitted to enlarge the boundaries of strictly geometrical knowledge, is that of which we proceed to treat.

Arithmetical Geometry, or the Application of Arithmetic to Geometry, is never employed for purposes of investigation or of analysis; its object is, in cases of particular lines, surfaces, and solids, to express by numbers their properties,—properties, the truth of which Geometry has already demonstrated, and for the numerical statement of which, Algebra has given the requisite formula, or method of operation. All questions, as they are called, in Mensuration, are instances of the application of Arithmetical Geometry.

A special example will show the respective provinces of Geometry, Algebra, and Arithmetic: we assert, that the square on the hypotenuse (h) of a right-angled triangle is equal the sum of the squares of the base (b) and of the perpendicular (a); Geometry demonstrates the truth of the assertion; Algebra, either investigates and analyzes that truth, or simply expresses it in the form of an equation, thus, $h^2 = a^2 + b^2$; and Arithmetic assuming a particular case, as that $h = 5$; $a = 4$; and $b = 3$, shows, that a square on the line of 5 equals the sum of the squares on the lines of 4 and 3 respectively; or that the square of 5 equals the sum of the squares of 4 and 3; 5×5, or 25, being equal to 4×4, or 16, added to 3×3, or 9. We shall afterwards see that no numbers, except 5, 4, and 3, or their equi-multiples, can with perfect accuracy give an arithmetical expression to the geometrical truth.

The main Axioms of Geometry, Algebra, and Arithmetic are the same: thus in Geometry we say,—Magnitudes equal to the same magnitude are equal to each other; in Algebra, Quantities are equal when each is equal to the same quantity; and in Arithmetic, Numbers are equal to one another when they are equal to the same number: but the three variations in the mode of stating the axiom are expressions for one and the same universal truth,—Things equal to the same thing are equal to one another.

On the ground that a perfectly accurate expression of all geometrical truths cannot be given by numbers, it is of great importance in reasonings strictly geometrical, where we seek for absolute truth, that we should not confound a Geometrical Demonstration with the Algebraical or Arithmetical representation of it. Being accustomed to speak of one magnitude as containing another smaller magnitude a certain number of times, we hesitate not to say, that a line contains 10 linear units; or a rectangle 25 square inches; or a solid 6 cubic inches: but Plane Geometry, having to do with space generally, does not reason respecting any definitely assigned quantity of space as expressed by concrete numbers, but about its universal properties.

A point, considered theoretically or mathematically, marks position and not magnitude, and no succession or series of such points could make up a line; and a line mathematical, having extension only in one direction through space, and not being a part of space, no succession of lines, that is of lengths without breadths, could form a surface; and a surface having extension only in two directions, length and breadth, no supposed piling up of surfaces could form a solid;—for the first surface in the imaginary series being without thickness, and the second and the third, &c., being equally destitute of that property, no number of such surfaces could form a solid. Solids, therefore, though bounded by surfaces, are not made up of surfaces; surfaces, though bounded by lines, are not made up of lines; neither are lines, though they have countless points in them, made up of an aggregation of points.

But *a point and a line considered practically*, and as they were considered before Geometry, or Land Measuring, became a theoretical Science, have in their very elements the property of extension; they do, and they must occupy space. On the Atomic

theory of Chemistry, there are ultimate atoms beyond which no actual division of an elementary body can proceed; so on the supposition that the points and lines employed in Mensuration possess visible properties, there are ultimate points and lines beyond which we cannot carry our process of refinement. In practice the point and the line are both visible, and whatever is visible ceases to be a mathematical point, or a mathematical line; it has length and breadth to make an impression on the optic nerve, and is therefore a surface.

The word *monads*, applied to denote the elementary atoms by the aggregation of which mineral, vegetable, or animal substances are formed, may be introduced, not disadvantageously, into Practical Geometry. A point is the monad, or element of a line; we may make it as small as we please; the thousandth part, or the ten-millionth part of an inch, but however small, it really possesses extension: and so a line may become thinner and thinner, and finer and finer, but it possesses visible properties of breadth,—in fact it is a surface; and the monad of a line, or its least elementary part, if visible at all, must also be a surface.

We need not continue an argument of this kind. When Arithmetic is applied to Geometry, the line is made up of points, or parts,—the point or part being some generally-recognised monad of length, or first and least element in the line: thus, when a line is made up of parts, each part containing one-tenth of an inch, then the unit, or monad of extension, is one-tenth of an inch. Again, for Surfaces made up of lines having visible breadth, the line that measures a surface in its length is made up of the units, or monads of length; and the line which measures the same surface in its breadth is made up of the same units, or monads, in this case called breadth. The monad of length and the monad of breadth, placed together at right angles to each other, constitute the monad or unit of Surface: it may be a square inch, or a square foot, or a square mile.

Thus 100th of an inch, taken in one direction, may represent the monad, or elementary part of length;

And 100th of an inch, taken in a direction at right angles from one extremity of the monad of length, will represent the monad of breadth;

And the monad of length and the monad of breadth, depth, or height,—for in a plane they are three words for one thing,—thus placed, constitute the monad or unit of surface.

Also, 100th of an inch, taken in a direction at right angles from the common point of junction of the two lines which represent the monad of surface, will represent the monad of thickness;

And the monad of length, the monad of breadth, and the monad of thickness thus placed, and constituting extension in three directions, originate the monad or unit of solidity.

Though, strictly speaking, the Monads, or least elementary parts of a line, are themselves lines; nay, from being visible, are actually surfaces; yet, if such a monad be set in motion, it generates or traces out a longer line; and a line thus formed and set in motion generates a surface; and a surface also being moved, traces out a solid.

It is on this principle that a visible point may cover from view the line made up of such points; the visible line cover the surface made up of such lines; and the surface cover from view the solid made up of such surfaces.

In Plane Geometry, however, we have to do only with surfaces and with lines as constituting the boundaries of surfaces: consequently, we have to consider only two dimensions—the measurement of length and the measurement of breadth, and the combination of the two as indicating the measurement of surface.

Lines and Surfaces, as well as all other magnitudes, may be expressed by numbers; indeed, without numbers they can only be very imperfectly expressed. To do this, as we have seen, some Standard of length, or of surface, is assumed; as, a linear inch, or a square inch: the number of such linear inches, or square inches, expresses the magnitude of the line or of the surface;

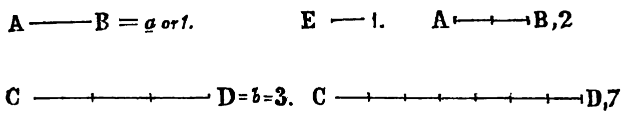
Thus the symbolical expression $\sqrt{2}$ denotes the diagonal of a square of which the side is 1: 4×3, or 12, the number of square units in a rectangle of which the sides contain 4 and 3 linear units respectively:

And generally, if we designate the number of linear units in one line by the algebraical symbol a, and the number in another line by b, the algebraical symbol ab will denote the area of the rectangle formed by the lines a and b.

We proceed, therefore, to consider the meaning which we should attach to algebraical and numerical symbols when applied to lines and surfaces.

FIRST,—*of a Line.*

Any straight line whatever, as AB, or a, or 1, may be taken to represent the unit of length; to any longer line, as CD, or b, or 3, we apply AB, or a, or 1, as the measuring line or unit; and if AB is contained an exact number of times, as 3 times, in CD, then we say, CD is equal to 3 times AB; thus $CD = 3\ AB = 3\ a = 1 \times 3 = 3$.



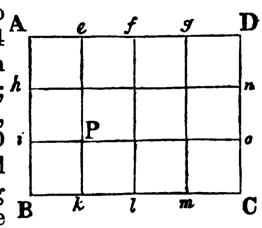
But AB may not be contained an exact number of times in CD, and consequently is not a measuring line for CD: we now seek for a third line as E; and if this third line is contained an exact number of times, as twice, in AB, and an exact number of times, as seven times, in CD,—then E is the common measuring line for both AB and CD, AB being equal to 2 E, and CD equal to 7 E. Thus $CD = \frac{7}{2}$ of $AB = \frac{7}{2}$ or $3\frac{1}{2}$.

Incommensurable lines will afterwards be treated of; but we have already ascertained that a line, any line, may be represented by a letter, as a, b, &c., and that for the letters may be substituted a number either integral, or fractional, or mixed.

SECOND,—*of the Rectangle, of the Square, and of the Parallelogram.*

The sides of the rectangle, and of the square, are divisible, we will suppose, into an exact number of linear units.

Take two lines at right angles to each other, $AB = 3$ linear units, $BC = 4$ linear units. By drawing lines through the points A, h, and i, parallel to BC; and through the points k, l, m, and C, parallel to AB; the rectangle ABCD will be divided into squares, each equal to $BiPk$, and thus each representing a square unit, an inch, or a foot, as the case may be: in the upper row, $AhnD$, there are four square units; in the second row, $hion$, 4 square units; and in the third row, $BioC$, also 4 square units. Now, there are as many



rows as there are linear units in AB, and as many squares in each row as there are squares on BC, each of which is equal to the square B*k*P*i*; therefore the whole number of square units in ABCD will be equal to the squares on BC, multiplied by the linear units in AB. If BC = 4 inches, feet, &c., and AB = 3 inches, feet, &c., the area of ABCD will contain 4×3, or 12 square inches, feet, &c.

Thus the product of two numbers, representing the linear units in each side of the rectangle, expresses the number of square units in the rectangle itself, and is an Arithmetical representation of the Area.

Of course this proof applies only to cases in which the two lines containing the rectangle have an exact measuring unit.

Now generalising what has been proved, and for 3 linear units in AB substituting the algebraical symbol a, and for 4 linear units in BC the symbol b,—the Area of any rectangle, as ABCD, will be represented by $a \times b$, or $a b$ square units; therefore $a b$ will be the algebraical symbol for the area of a rectangle; *i.e.*, the units in the altitude multiplied by the units in the base, will give the square units in the whole rectangle.

"Hence it follows," says Potts (*Elements of Geometry*, p. 68), "that the term *rectangle* in Geometry corresponds to the term *product* in Arithmetic and Algebra; and that a similar comparison may be made between the products of the two numbers which represent the sides of rectangles, as between the areas of the rectangles themselves. This forms the basis of what are called Arithmetical or Algebraical proofs of Geometrical properties."

When the altitude and the base are equal, the surface is a square; and thus, a being equal to b, the square on a line is represented by $a\,a$, or a^2.

And as Parallelograms, of which Rectangles are one kind, upon the same base and of the same altitude are equal in Area, the *Area of any Parallelogram* may be found by multiplying the altitude into the base.

THIRD,—*of the Triangle and all other Rectilineal Figures.*

The Area of the triangle is equal to half the area of a parallelogram on the same or on an equal base, and between the same parallels; and we may therefore represent the *Area of the Triangle* by taking half the area of the rectangle which has the

same base and altitude as the triangle: thus, if $a b$ is the algebraical expression for the area of the rectangle, $\frac{1}{2} a b$, or $\frac{a b}{2}$, will be the algebraical expression for the area of the triangle.

And, inasmuch as all rectilineal figures may be resolved into triangles by diagonals from the angular points, we are able to express *the area of any rectilineal figure* by finding the area of each triangle separately, and then adding the areas together; $\frac{a b}{2} + \frac{a' b'}{2} + \frac{a'' b''}{2}$, &c., according to the number of triangles into which the rectilineal figure is divided, will give the whole area. On this plan the altitude of each separate triangle into which the figure is divided must be taken.

FOURTH,—*of the Regular Polygon and of the Circle.*

If from the centre of a regular polygon lines be drawn to the angular points, the polygon will be divided into equal triangles; and if the area of one of the triangles be found, that area multiplied by the number of triangles will give the *Area of the Regular Polygon.*

When we suppose a regular polygon to have an infinite number of sides, the sum of the sides will be equal to the circumference of a circle, the radius of which equals the altitude of each triangle in the polygon; and when that circumference is spread out into a straight line, it will represent the base of a triangle of which the altitude equals the radius of the circle; therefore the *Area of a Circle* will equal half the circumference multiplied by the radius.

The diameter and circumference of a circle are incommensurable; but it has been ascertained that when the diameter is 1, the circumference is 3·1415926 &c.; and on this fact the methods are founded for calculating various problems in the Mensuration of Circles.

Those who desire to study more completely the Application of Algebra to Geometry, are referred to Waud's "*Treatise on Algebraical Geometry*," or to Bourdon's "*Application de l'Algèbre à la Géométrie;*" and for the Application of Arithmetic to Geometry, they are recommended to "*A Treatise on Mensuration*," and especially to the "*Appendix*," by the Commissioners for National Education in Ireland;—there is here a thoroughly scientific explanation of the Principles of Mensuration. For useful and improving exercises, the learner will find valuable assistance in "*Mensuration, Plane and Solid*," by the Rev. J.

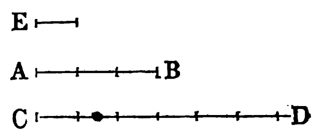
Sidney Boucher, M.A. The work is well described as a "Series of Arithmetical Illustrations of the most important Practical Truths established by Geometry."

SECTION VI.

ON INCOMMENSURABLE QUANTITIES.

When of two lines the less does not measure the greater, we may, as we have seen (p. 23), often find a third line that will be a *Common Measure* of both; such magnitudes are *Commensurable;* but when lines, or magnitudes, are so related, that, though one is capable of being represented in the terms of a certain unit, the other is not,—those lines, or magnitudes, are *Incommensurable*, and are to be explained by the Principles on which the Theory of incommensurables in Arithmetic is founded:

Thus in the lines AB and CD, if, when AB is measured by a third line E, CD is not measured by that third line, the lines are incommensurable.



An instance of Incommensurable quantities may be given from the number 2; there is no number either integral, or fractional, or mixed, which will express or measure the exact square root of 2. We may indeed employ the symbolical representations $\sqrt{2}$, or $\sqrt{3}$, or $\sqrt{50}$, &c., and we may reason about them, just as if their exact numerical values existed; but the numerical square root of such numbers we can never reach. In the same way there are many common fractions which cannot be expressed with perfect accuracy as decimal fractions: take the common fraction $\frac{4}{7}$, and attempt to find its exact value as a decimal fraction; we obtain ·571428571428, &c., *ad infinitum*, but never reach the absolutely true decimal representation of $\frac{4}{7}$.

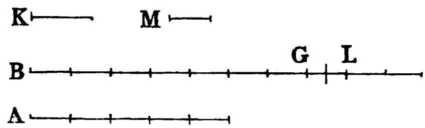
Among instances of incommensurable quantities or magnitudes, we may select the Diagonal and the Side of a Square. It is a Geometrical Principle, that the sum of the squares of the sides about a right angle is equal to the square on the side opposite the right angle. Take a square having 10 for its side; the square on one side is equal to 100; the square on the other side also equal to 100; and $100+100=$ the square on the diagonal:

but 200 has no exact square root; we express it symbolically and say $\sqrt{200}$ = the diagonal of a square of which the side = 10, but the representative numerical value of $\sqrt{200}$ cannot be found.

It is for a reason of this nature that in many instances the Arithmetical Illustration of a Geometrical Truth cannot be substituted for the Geometrical Demonstration; for the former in such instances only approximates to the Truth, the latter is the absolute Truth itself.

But the *Approximation* may be carried out to any assigned degree of exactness; if we are not satisfied with being so near the exact truth as the thousandth part of a unit, we may diminish the possible error to within the millionth, or billionth part, according to our fancy or the necessities of the case.

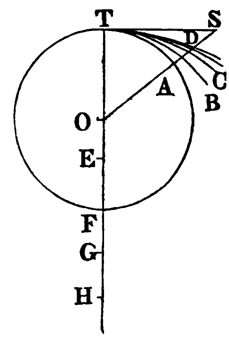
The proof of this we subjoin from the Article, "*Incommensurable*" in the *Penny Cyclopædia*, Vol. XII, p. 456.



"Let A and B be two incommensurable magnitudes; and let K be a third magnitude of the same kind, which may be as small as you please, only that it be given and known. Now, some aliquot part of A must be less than K; if not the hundredth, try the thousandth; if not the thousandth, try the millionth; and so on. Whatever K may be, it is possible to divide A into equal parts, each of which shall be less than K. Let M be such an aliquot part of A; and having divided A into its parts, set off parts equal to M along B. Then A and B being incommensurable, B will not contain M, the measure of A, an exact number of times, but will lie between two multiples of M, say B G and B L. From this it is obvious that B does not differ from either B G or B L by so much as G L, and therefore not so much as K. But B G and B L are both commensurable with A, since all three are multiples of M. Here there are B G and B L, the first a little less than B, and the second a little greater, neither differing from B by so much as K, but both commensurable with A. Thus it is also evident, that two whole numbers may be found which shall be as nearly as we please in the same ratio as two incommensurable quantities.

"The difficulty thus inherent in the application of arithmetic to concrete magnitude is not met with in practice, because no

case can arise in which it is necessary to retain a magnitude so closely that no alteration however small can be permitted. But in exact reasoning, where any error however small is to be avoided, it is obvious that the arithmetic of commensurable magnitudes, and the arithmetic (if there be such a thing) of incommensurable magnitudes, must not be confounded."



A clear example *of approximating nearer and nearer to a point which can never be reached*, is supplied by drawing a tangent to the diameter of a circle, and from the centre drawing a line until it shall cut the tangent,—this line cutting the tangent being named the *secant;* as in the circle TAF, TS being the tangent, and OS the secant. If in TF, or TF produced, points be taken, E, F, G, H, &c., as centres of circles, to all of which TS shall be the common tangent; then each circle, as it cuts AS, shall approach nearer to S, as in the points B, C, and D; but no circle shall ever pass through the point S, inasmuch as TS being the common tangent, the circles cannot touch TS in any point except the point T; for if they did, a curved line and a straight line would coincide.

In a similar way, if £20 be subscribed annually for the purchase of books, and at the end of each year the books of the previous year be sold at half-price, and the proceeds of the sale added to a new subscription of £20, and if the two sums thus added be expended in the purchase of more books, and so on, year after year, the half of the value of the preceding year's books being yearly added to the fixed subscription of £20, there will never be £40 worth of books purchased in any one year; though in each successive year the expenditure will approach nearer and nearer to £40.

SECTION VII.

OF WRITTEN AND ORAL EXAMINATIONS.

Written Examinations may be considered as the suitable test of accuracy,—oral examinations of readiness; the one allows of the exercise of the reflective powers; the other brings into play quickness of perception and leads to promptness of action. The union of the two kinds, according to the nature of the subject under examination, should be aimed at in striving to ascertain progress, and actual knowledge and skill.

The advantages of Written Examinations in Geometry and in kindred subjects, are well pointed out by S. F. Lacroix in his Essay, "*On Teaching in general, and on the Teaching of Mathematics in particular.*" He says, p. 197, 198, "It has been proposed to substitute examination by writing, which gives to the candidate more time to collect his ideas,—which lessens the disadvantages of timidity,—and which being carried on at the same time for all the pupils, permits the same questions to be asked of each, and renders their answers more suitable for comparison. This written examination may also be less troublesome for the Examiner, because, instead of the unremitting attention which he must give to oral answers, and the efforts of memory necessary to recall to his mind the impression which those answers make, he has only a labour capable of being divided and suspended when he experiences too much fatigue; and all the papers which serve as a basis for his judgment, are at the same time under his eye."

"It is principally on the applications of theories, that the questions of a written examination ought to run, and on calculations, altogether out of place in an oral examination."

For subjects not mathematical, however, a high place may be assigned to Oral Examinations. Of written examinations Lacroix afterwards says, p. 199, "But by this *written* examination alone we are never perfectly informed as to the readiness with which a scholar may express himself,—a readiness which it is necessary to exercise and encourage, because it is useful at almost every moment of life, and because it is indispensable for men who will some day have projects to bring forward or to discuss in the presence of their companions or of their superiors, and it is only an oral examination which can make them appreciated in this respect."

The Advantages of Written Examinations in Geometry have suggested the "Skeleton Propositions;" and these advantages will, it is hoped, be increased by the aid which such outline propositions afford for training to method and exactness. Lest, however, the assistance given, by placing references in the margin, should be greater than is good for more advanced learners, *a Second Course of Examinations* is recommended,—if indeed it be not absolutely necessary; a Course in which no other aid is afforded to those under examination than the General Enunciations of the Propositions and a few vertical lines, within which learners are themselves to place the references to the truths already established, and on which the construction and the demonstration depend.

For those who purchase only the "Gradations of Euclid," and who yet wish to know the Plan proposed for the "Pen and Ink Examinations," an Example is now added of both Series,—of the one that has the references printed in the margin, and of the other without any references.

First,—with all the references by aid of which the Examination is to be conducted:

PROP. 1.—PROB.

To describe an equilateral triangle upon a given finite straight line.

SOLUTION.—Psts. 3 and 1.

DEMONSTRATION.—Def. 15, and Ax. 1.

EXP.	1	Datum.
	2	Quæs.
CONS.	1	Pst. 3.
	2	" 3.
	3	" 1.
	4	Sol.
DEM.	1	C 1 & Def. 15
	2	C 2 & Def. 15
	3	D 1,2 & Ax. 1
	4	D. 1, 2, 3.
	5	Recap.

USE AND APPLICATION.

Second,—without any of the references, to make the examination strict and thorough.

PROP. 1.—PROB.

To describe an equilateral triangle upon a given finite straight line.

SOLUTION.

DEMONSTRATION.

EXP.

CONS.

DEM.

USE AND APPLICATION.

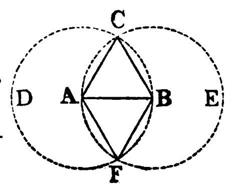
The Skeleton Proposition when filled up will appear as below, symbols and contractions being allowed.

Prop. 1.—Prob.

To describe an equilateral triangle on a given finite straight line.

Solution.—Psts. 3 and 1.—Pst. 3. A $\odot$ may be described from any centre at any distance from that centre.
Pst. 1. A line may be drawn from any one point to another.

Demonstration.—Def. 15, and Ax. 1.—Def. 15. A $\odot$ is a plane figure bounded by one continued line called its $\odot$ce, and having a certain point within it from which all st. lines drawn to the $\odot$ce are equal.
Ax. 1. Magnitudes which are equal to the same, are equal to each other.



Exp.	1	Dat.	Given, the st. line AB;
	2	Quæs.	on it to desc. an equil. $\triangle$
Cons.	1	Pst. 3.	From centre A with AB desc. $\odot$ BCD;
	2	„ 3.	from centre B with BA desc. $\odot$ ACE;
	3	„ 1.	Draw AC and BC;
	4	Sol.	then $\triangle$ ABC is the eq. lat. $\triangle$ required.
Dem.	1	C.1, Def.15	$\because$ A is centre of $\odot$ BCD, $\therefore$ AC = AB;
	2	C.2, Def.15	$\because$ B „ $\odot$ ACE, $\therefore$ BC = AB:
	3	D.1,2,Ax.1	But AC, BC each = AB, $\therefore$ AC = BC.
	4	D. 1, 2, 3.	and $\therefore$ AB = AC = BC;
	5	Recap.	therefore $\triangle$ ABC is equil. and on AB.

Q.E.F.

Use and Application.—This problem may be applied to the measurement of inaccessible lines, by drawing on wood or brass an equil. triangle, and using the instrument, by placing it at A, and along the line AB, so that C and B may be seen; then if it be carried along AB, until at B, C and A can be seen along the edges of the instrument, the side AB will have been traversed, and AB is equal to AC or BC. The AB, being measured, will equal the other distances, AC, or CB.

PART I.

GRADATIONS IN EUCLID.

ELEMENTS OF PLANE GEOMETRY.

BOOKS I. & II.

GRADATIONS IN EUCLID.

BOOK I.

DEFINITIONS.

1. A *Point* is that which has no parts, or which has no magnitude: it marks position.—THEON and PYTHAGORAS.

"A point is that of which there is no part."—EUCLID: or, "which cannot be parted or divided."—PROCLUS.

"A point is a monad having position."—PYTHAGORAS. A mathematical point cannot be drawn; for a visible point is, in fact, a surface.

2. A *Line* is length without breadth; or it is extension in one direction.

A mathematical line cannot be drawn; for whatever is visible must have breadth.

A line is *measured* by the number of units, or monads, of length contained in it;—as, 5 inches; 9 feet; 13 miles.

3. The *extremities*, or *ends*, of a line are points.

4. A *right*, or *straight line*, is that which lies evenly between its extreme points.

"A straight line is the shortest distance between two points."—ARCHIMEDES, *adopted by* LEGENDRE.

A straight line is that of which the extremity hides all the rest, the eye being placed in the continuation of the line."—PLATO. Plato's line was thus a visible, not a mathematical line.

5. A *superficies*, or surface, is that which has only length and breadth; it is extension in two directions.

6. The *extremities*, or *boundaries*, of a surface, are lines.

7.—A *plane surface* is that in which any two points being taken, the straight line joining them lies wholly in that surface. —HERO THE ELDER.

"A plane surface is that which lies evenly, or equally, with the straight lines in it."—EUCLID.

"A plane surface is one whose extremities hide all the intermediate parts, the eye being placed in its continuation."—PLATO. "A plane surface is the smallest surface which can be contained between given extremities." "A plane surface is that to which a straight line may be applied in any manner of way."

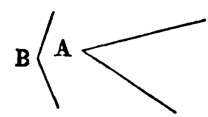
A plane surface is *measured* by the number of square units, or monads, of surface, contained within its boundaries;—as, 4 square inches; 9 square feet; 13 square yards, &c.

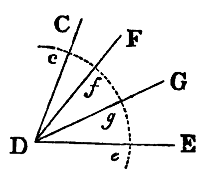
8. A *plane angle* is the inclination of two lines to each other in a plane, which meet together in the same point, but are not in the same straight line.

A plane angle is the opening of two lines from their point of meeting, or of intersection.

9. A *plane rectilineal angle* is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

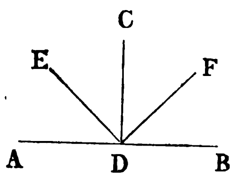
A plane rectilineal angle is the opening of two straight lines from their point of meeting or of intersection, that point being the vertex. The magnitude of the angle is altogether independent of the length of the lines; angle B is greater than angle A. Where there are several angles meeting at a point, each angle is distinguished by three letters, the letter at the point of meeting of the lines forming the angle, being named in the middle; as, CDF, FDG, GDE.





A plane rectilineal angle is *measured* by the number of degrees in the arc of a circle, of which the centre is at the angular point, and the circumference cuts the lines forming the angle: thus, the arc *e c* is the measure of the angle EDC; arc *eg*, of angle EDG, &c. A degree is the 360th part of the circumference of a circle.

10. When a straight line standing on another straight line makes the adjacent angles equal to each other, each of these angles is called a *right angle;* and the straight line which stands on the other is called a *perpendicular* to it.



The angle CDA being equal to angle CDB, each of them is a right angle, and CD is a perpendicular to AB.

The *measure* of a right angle is always equal to an arc of 90°, *i. e.*, to the fourth part of the circumference of a circle: thus, if from D an arc were drawn cutting the lines radiating from D, the arc BC would measure the angle BDC, and the arc AC the angle ADC.

11. An *obtuse angle* is an angle greater than a right angle; as angle EDB.

The *measure* of an obtuse angle is always greater than 90°.

12. An *acute angle* is an angle less than a right angle; as angle FDB.

The *measure* of an acute angle is always less than 90°.

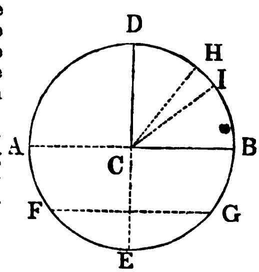
13. A *term*, or *boundary*, is the extremity of any thing.

A boundary is the limit within which anything is contained.

14. A *figure* is a surface enclosed by one or more boundaries.

"When all the points in a figure are also points in the same plane, the figure is called a plane figure."—HOSE.

15. A *circle* is a plane figure contained by one line, which is called the *circumference*, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.



A circle is traced out by the motion of a line round one of its extremities: if the line BC revolve round the extremity C, the other extremity B will trace out the circumference of a circle. Any portion of a circumference is named *an arc*.

A circumference is *measured* by being divided into 360 equal parts, each part being called a degree. When the circumference is thus divided, and lines are drawn from the centre to each of these divisions, the angle formed by every adjacent pair of lines, is called an angle of one degree. "We may thus compare angles together by comparing the number of degrees contained in the intercepted arcs of a circle described from the angular point as a centre." Thus if the arc BD contains 90°, and the arc BH 45°, the angle BCD will be double of the angle BCH.

16. And the point (from which the equal lines are drawn) is

called the *centre* of the circle; as, C is the centre of the circle ADBE, having the lines CA, CD, CB, &c., all equal.

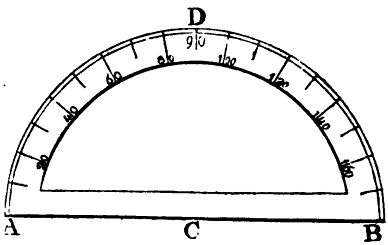
The straight lines from the centre to the circumference are *radii*.

17. A *diameter* of a circle is any straight line drawn through the centre, and terminated both ways by the circumference; as AB, or DE, in the circle ADBE.

A diameter is double of a radius.

18. A *semicircle* is the figure contained by the diameter and the part of the circumference cut off by the diameter; as ADBCA.

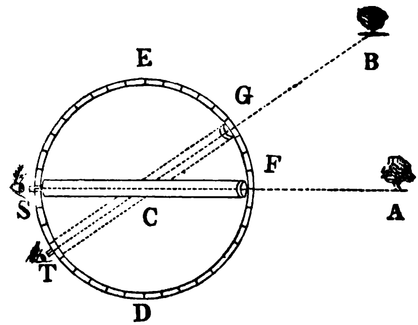
The centre of the diameter forming one boundary of the semicircle, is the same with the centre of the circle; and the semi-circumference being divided into 180 equal parts, and lines from the extremities of those equal parts being drawn to the centre, the arc opposite to each angle thus formed, will be *the measure of an angle of one degree.*



The *Semicircle* thus divided, (or the *Protractor*, as it is called when the lines from the centre C are drawn to the edges of a rectangular figure), is employed *for the measurement of angles*, by laying the centre C at the angular point, and the edge CB, or AB, along one line of the angle, and then noting under what degree the other line passes: the arc between the two lines is the measure of the angle. Or, if it be required *to draw an angle of a fixed number of degrees*, the semicircle thus divided will enable us to do it; for, draw a line, and determine where the angular point in it is to be,—then apply to that point the centre C, and lay the edge CB, or CA, along the line already drawn; note by a mark the degrees required, and join that mark to the angular point C by a straight line, the two lines will enclose the angle required.

N.B. On the principle that heavy bodies, near the earth's surface, when left free to move, gravitate towards the earth's centre, the semicircle or a skeleton of it called a level, may be employed with a plummet suspended from D, or 90°, to *ascertain both the horizontal and vertical lines*: if the string by which the plummet is suspended, covers the line at C, AB is in the horizontal line, and a line from D to C, or the string itself, will be the vertical line. An instrument of this kind is made use of by builders and others, and for purposes of levelling, &c.

In Surveying, *for the measurement of angles from any point to distant objects*, several instruments have been constructed: as the Sextant, an arc of 60°; the Quadrant, an arc of 90°; and the Theodolite, a whole circle of 360°, divided into two semicircles of 180° each.



The Theodolite has revolving on its centre C, a graduated index, on which is fixed a telescope SCF, with fine cross wires on the object glass F. There are also spirit levels attached, for ascertaining the true horizontal line. The observer sets the telescope SCF, so that an object at A may be seen from S, through the telescope; the theodolite remaining fixed, the telescope is turned round until from T the second object B can be covered by the intersection of th wires of the object-glass of the telescope: the number of degrees traversed by the telescope from S to T, or from F to G, is noted, and that number is the measure of the angle ACB.

19. A *segment* of a circle is the figure contained by a straight line and the part of the circumference which it cuts off; as in Def. 15, the figure FGEF.

In a segment the straight line, as FG, is the *chord;* the part of the circumference cut off, as FEG, the *arc.*

A *Sector* is any portion of a circle bounded by two radii and the arc which those radii intercept, as BCDB, BCIB.

20. *Rectilineal figures* are those which are bounded by right or straight lines.

21. *Trilateral figures*, or *Triangles*, are bounded by three straight lines.

All other rectilineal figures may be resolved into triangles, by joining some one angular point and the other angular points.

22. *Quadrilateral figures* are bounded by four straight lines.

The *Diagonals* of a quadrilateral are the lines joining the opposite angles.

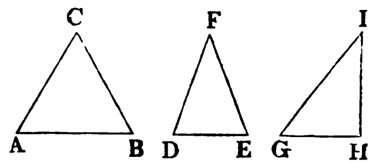
23. *Multilateral figures*, or *Polygons*, are bounded by more than four right lines.

They are named from the number of angles: as, *pentagon*, a figure of five angles, or sides; *hexagon*, of six; *heptagon*, of seven; *octagon*, of eight, &c.

24. An *Equilateral Triangle* is that which has three equal sides; as, ABC.

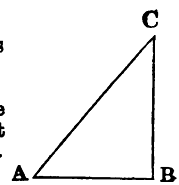
25. An *Isosceles Triangle* is that which has two equal sides, or legs; namely, DF and EF.

26. A *Scalene Triangle* is that which has three unequal sides; as, GHI.



27. A *Right-angled Triangle* is that which has a right angle; as, ABC.

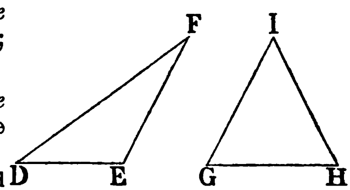
The side opposite the right angle, AC, is named the *hypotenuse*, or subtend; of the sides about the right angle, AB is named the *base*, and BC the *perpendicular*.



28. An *Obtuse-angled Triangle* is that which has an obtuse angle; as, DEF.

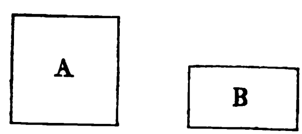
29. An *Acute-angled Triangle* is that which has three acute angles; as, GHI.

The right, and the obtuse-angled triangles, have also two acute angles.



30. Of *quadrilaterals*, that is, of four-sided figures, a *square* has all its sides equal, and all its angles right angles; as A.

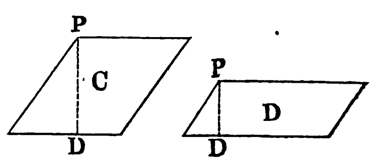
31. An *oblong* is a figure which has all its angles right angles, but not all its sides equal; as B.



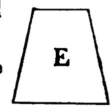
The oblong is the same as the *rectangle* of Book II.

32. A *rhombus* has all its sides equal, but its angles are not right angles; as C.

33. A *rhomboid* has its opposite sides equal to each other, but all its sides are not equal, nor are its angles right angles; as D.

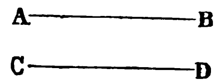


The term *parallelogram* may supersede that of rhomboid.
In C and D, the perpendicular PD is the *altitude* of the figure.



34. All other four-sided figures, as E, are called *Trapeziums.*

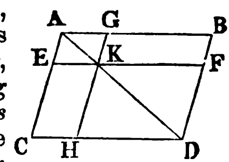
The name of the figure is derived from the shape of the Greek tables, at which the master sat at the broad end, that he might be seen by all his guests.



35. *Parallel straight lines* are such as are in the same plane, and which being produced ever so far both ways, do not meet; as AB, CD.

The least objectionable of the many definitions proposed, from which we may reason respecting parallel lines, is that which Potts says (p. 50), "simply expresses the conception of equidistance;" thus, "Parallel lines are such as lie in the same plane, and which neither recede from, nor approach to, each other."

In the first Six Books of Euclid, all the lines are supposed to be in the same plane; the *test of parallelism* is, that two lines, being in the same plane, never meet, though indefinitely produced.



A. *A parallelogram* is a four-sided figure, of which the opposite sides are parallel, as AB to CD, and AC to BD. The *Diameter*, or diagonal, AD, is the straight line joining two opposite angles: the *parallelograms about the diagonals* AEKG and KHDF, are those through which the diagonal passes; and the *Complements* are the two parallelograms, ECHK and GKFB, through which the diagonal does not pass.

Any figure with an equal number of equal sides, as four, six, eight, &c., will have its opposite sides parallel; but in the *Elements of Euclid* the name parallelogram is restricted to four-sided figures.

Observation.—"It is necessary to consider a solid,—that is, a magnitude which has length, breadth, and thickness,—in order to understand aright the definitions of a point, a line, and a superficies. A solid, or volume, considered apart from its physical properties, suggests the idea of the surfaces by which it is bounded; a surface, the idea of the line or lines which form its boundaries; and a finite line, the points which form its extremities. A solid is therefore bounded by surfaces; a surface is bounded by lines; and a line is terminated by two points. A point marks position only; a line has one dimension, length only, and defines distance; a superficies has two dimensions, length, and breadth, and defines extension; and a solid has

three dimensions, length, breadth, and thickness, and defines some definite portion of space."

"It may also be remarked, that two points are sufficient to determine the position of a straight line; and three points not in the same straight line are necessary to fix the position of a plane."—*Pott's Euclid*, p. 44.

Postulates.

1. Let it be granted that a straight line may be drawn from any one point to any other point:

2. That a terminated straight line may be produced to any length in a straight line:

3. And that a circle may be described from any centre at any distance from that centre.

The first and second Postulates concede the use of a ruler, but not of a scale; the third, that of the compasses,—but not that a circle can be described round a given centre with a radius, or distance in the compasses of a given length.

Euclid himself gave *three other postulates*, which modern Editors of the *Elements* place as the tenth, eleventh, and twelfth Axioms. Those postulates were—4. Let it be granted that two straight lines cannot enclose a space: 5. That all right angles are equal: and 6. That when two lines are met or crossed by a third line, so that the two exterior angles on the same side of it taken together are less than two right angles, the two lines so crossed shall meet, if continually produced.—*Smith's Biographical Dictionary*, '*Eucleides*,' Vol. II., p. 66.

Axioms.

I. The Seven Axioms, which apply to number and quantity as well as to magnitude, were called by Euclid Common Notions. They are—

1. Things which are equal to the same thing, are equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be taken from equals, the remainders are equal.
4. If equals be added to unequals, the wholes are unequal.
5. If equals be taken from unequals, the remainders are unequal.
6. Things which are double of the same, are equal to one another.

7. Things which are halves of the same, are equal to one another.

II. The Five Axioms, which apply especially to magnitude, are peculiarly Axioms of Geometry:

8. Magnitudes which coincide with one another,—that is, which exactly fill the same space,—are equal.

"This Axiom is properly the definition of Geometrical equality."

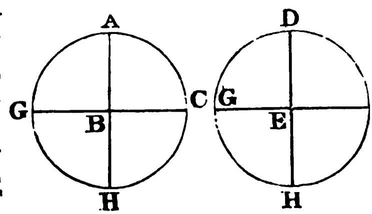
We prove the equality of two straight lines by placing them one upon the other, or by conceiving them so placed, and ascertaining that the extremities coincide: we prove the equality of two angles, by placing vertex on vertex, and line on line, and then if the openings between the lines are equal, the other line of the one angle will coincide with the other line of the second angle, and the angles are equal.

And so with respect to any figure: if all the parts and boundaries of one figure fall upon and cover all the parts and boundaries of another figure, the two figures are equal. The name of *Super-position* is given to this process, which may be either actual or mental:—the one, when performed, being the proof to the senses; the other, though only conceived to be done, being demonstration to the mind.

9. The whole is greater than its part.

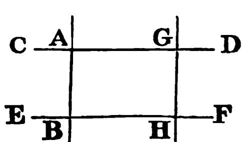
10. Two straight lines cannot enclose a space.

11. All right angles are equal to one another.



This may be illustrated by supposing that equal circles have been drawn from B and E, the vertices of the right angles: the fourth part of each circle, ABCA, or DEFD, or the arcs CA, FD will be the measure of each right angle; but the circles AGHC and DGHF are equal; therefore the fourth parts of them must be equal; and consequently the angles measured by those equals must themselves be equal.

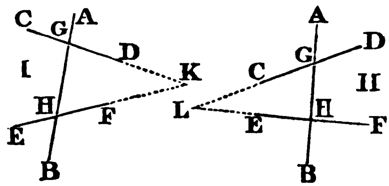
12. If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these two straight lines, being continually produced, shall at length meet upon that side on which the angles are less than two right angles.



This is almost equivalent to saying, that if two lines are parallel, all the perpendiculars enclosed between them shall be equal; for if CD

is parallel to E F, and the perpendicular G H less than the perpendicular A B, the lines C D and E F are nearer together towards D and F than towards C and E, contrary to the definition of parallel lines.

An illustration, by figures I. and II., will make the axiom plainer. I. The angles D G H and G H F are less than two right angles, and C D, E F meet in K. II. The angles C G H and G H E, being less than two right angles, the lines C D and E F meet in L; *i. e.*, on the side of A B on which the angles are less than two right angles.



Lines, like C D and E F (in I.) *converge*, when they approach nearer and at last meet in a point K: but lines, like L C D and L E F (in II.) *diverge*, when setting out from a point, L, they recede more and more.

When one line, A B, meets two other lines, C D and E F, there are four angles formed on the right hand of A B, and four on the left: of these angles, the *interior angles* are C G H, G H E, D G H, and G H F; the *exterior angles* are A G C, A G D, B H E, and B H F; the *opposite angles* are A G C to A H E, or B H F to B G D, &c.; the *adjacent angles*, A G C and A G D, or A G D and D G H, &c.; the *vertical angles*, those of which the vertex is at the same point, as C G H and A G D; and the *alternate angles*, every other one, as C G H and G H F, or D G H and G H E.

Instead of the twelfth axiom, Playfair adopts the following:—"Two straight lines which intersect one another, cannot be both parallel to the same straight line."

PROPOSITIONS.

Prop. 1.—Prob.

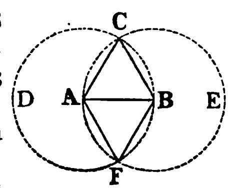
To describe an equilateral triangle upon a given finite straight line.

Solution.—Pst. 3. A ⊙ may be described from any centre at any distance from that centre.

Pst. 1. A straight line may be drawn from any one point to any other point.

Demonstration.—Def. 15. A ⊙ is a plane figure contained by one line which is called the ⊙ce, and is such that all straight lines drawn from a certain point within the figure (called the centre) to the ⊙ce are equal to one another.

Ax. 1. Magnitudes which are equal to the same magnitude, are equal to one another.



Exp.	1	Datum.	Given, the st. line AB;
	2	Quæs.	on it to desc. an equil. △
Cons.	1	Pst. 3.	From centre A with AB desc. ⊙ BCD;
	2	„ 3.	and from centre B with BA desc. ⊙ ACE;
	3	„ 1.	from the point C where the circles cut, draw CA and CB;
	4	Sol.	then △ ABC shall be the eq. lat. △ required.
Dem.	1	C.1, Def.15	∵ A is centre of ⊙ BCD, ∴ AC = AB;
	2	C.2, Def.15	∵ B „ ⊙ ACE, ∴ BC = BA:
	3	D.1,2,Ax.1	But AC, BC each = AB, ∴ AC = BC.
	4	D. 1, 2, 3.	Thus AB, BC, and AC, sides of a triangle, are equal to one another.
	5	Recap.	*Therefore the* △ ABC *is equil. and on* AB.

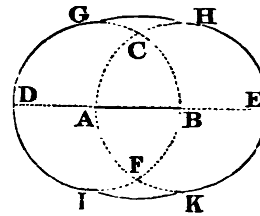
Q.E.F.

Scholium.—A second equil. triangle, AFB, may be drawn on the other side of AB.

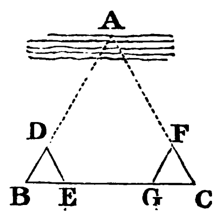
Use and Application.—1. The only use to which Euclid applies this proposition, is in solving the next two problems, and problems 9, 10, and 11.

2. For merely practical purposes it is sufficient, in describing an equil. triangle, to draw arcs intersecting in the common point C, or F, and to join the points A, B, C. By an accommodation of this method, an isosceles

triangle may be made: thus,—from the extremities of the base draw arcs with a common radius equal to the equal sides of the triangle, and join the points A, B, and C.



3. A figure approximating to an *oval* may be drawn by describing, from the extremities of a given line A B, equal circles intersecting in C and F, and by taking the diameter A E, or B D, and from C and F drawing the arcs G H and I K to meet the arcs I G and K H; the figure D G H E K I D will approximate to an oval.



4. By drawing an equilateral triangle on wood or brass, an instrument, B D E, or C F G, may be made, with which *an inaccessible distance can be measured.* Let A be an object on the other side of a river; at station B place the instrument so that A can be seen along the edge B D; then without changing the position of the instrument, look along the other edge B E, and set out a line of indefinite length as a continuation of B E; carry the instrument along the line B C, with its edge B E upon the line B C, until by looking along the edge D E brought to coincide with C F, the same object A will appear in a straight line with C F: then A, B, and C will be the angular points of an equilateral triangle; and as the sides are equal, by measuring from the station B to the station C, the distance from B to A, or from C to A, will be ascertained.

Prop. 2.—Prob.

From a given point to draw a straight line equal to a given straight line.

Solution.—Pst. 1. A st. line may be drawn from any point to any other point. P. 1. On a given line to desc. an equil. triangle.

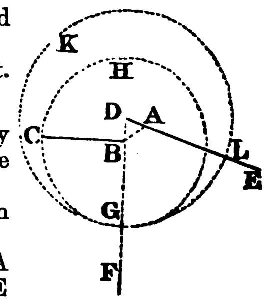
Pst. 2. A terminated st. line may be produced to any length in a st. line.

Pst. 3. A circle may be drawn from any centre at any distance from that centre.

Demonstration.—Def. 15. All st. lines from the centre of the circle to the circumference, are equal.

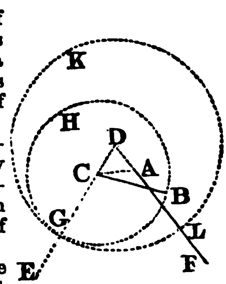
Ax. 3. If equals be taken from equals, the remainders are equal.

Ax. 1. Things which are equal to the same thing, are equal each to the others.



Exp.	1	Data.	Given the point A, and the st. line BC;
	2	Quæs.	from A to draw a st. line = BC.
Cons.	1	Pst. 1.	From B the extremity of BC, draw a line to the point A;
	2	P. 1.	on AB construct an equil. △ BDA;
	3	Pst. 2.	lengthen DB and DA indefinitely to E and F;
	4	Pst. 3.	from centre B with BC desc. the ⊙ CHG;
	5	"	and from centre D with DG desc. ⊙ GLK;
	6	Sol.	then the st. line AL from A = the given line BC.
Dem.	1	C. 4, 5.	∵ B is the centre of ⊙ CGH, and D of ⊙ GLK;
	2	Def. 15.	∴ BG = BC, and DL = DG.
	3	C. 2.	But the line DA = DB;
	4	Sub.	and taking away these equals from the equals DL and DG,
	5	Ax. 3.	the rem. AL = the rem. BG:
	6	D.2, Ax. 1	But BG being = BC, ∴ the rem. AL = BC.
	7	Rec.	*Wherefore from the given point* A *has been drawn a st. line* AL = *the given st. line* BC.

Q.E.F.



Scholium.—When the given point is out of the given line, or of the given line produced, this problem admits of eight cases, each of which is a solution of the problem; but if the given point is in the given line, or in the given line produced, of only four cases.

It is very conducive to the learner's improvement, when the proposition admits of it, to vary the mode of solution: of the eight cases mentioned, we will take another, in which the given point A is joined to C, the other extremity of the line BC.

The same method will be pursued in the Solution: join A and C, and on AC construct an equil. triangle ADC; produce its sides to

E and F; and with CB as radius desc. the circle GBH, and with DG as radius, the circle GLK; the line AL will be drawn equal to CB.

The Demonstration follows the same course as in the first case given above. The learner may solve some of the other cases for himself.

Use and Application.—Practically the given distance BC will be taken in the compasses, or measured by a string, or some instrument, as a foot, or a yard, and a line of the required length be marked off from A, as AL.

—

Prop. 3—Prob.

From the greater of two given lines to cut off a part equal to the less.

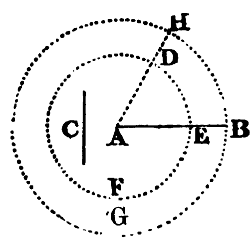
Solution.—P. 2. From a given point to draw a line equal to a given line.

Pst. 3. A ⊙ may be described from any centre at any distance from that centre.

Demonstration.—Def. 15. All straight lines from the centre to the circumference of a ⊙, are equal.

Ax. 1. Magnitudes which are equal to the same magnitude, are equal to each other.

Exp.	1	Data.	Given the st. lines AB and C, AB being the greater of the two;
	2	Quæs.	it is required to cut off from AB a part = the less C.
Cons.	1	P. 2.	From the point A draw a st. line AD = to C;
	2	Pst. 3.	and from A with radius AD describe the ⊙ DEF;
	3	Sol.	then the st. line AE cut off from AB, = C the less.
Dem.	1	C. 2.	∵ A is the centre of the ⊙ FED,
	2	Def. 15	∴ the st. line AE = the st. line AD:
	3	C. 1.	But the st. line AD = the given st. line C,
	4	Ax. 1.	consequently, the st. line AE, cut off from AB, = the st. line C.
	5	Rec.	*Therefore from* AB, *the greater has been cut off,* &c. Q.E.F.

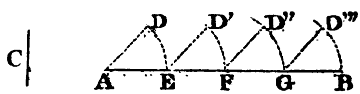


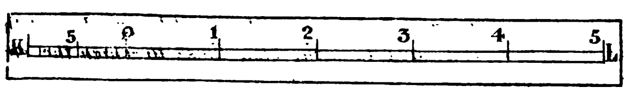
SCHOLIUM.—A less line AD may be made equal to a greater AB, by describing a circle GBH with the radius AB, and producing AD until it meets the circle in H; then AH will equal AB.

USE AND APPLICATION.—1. This problem is performed *practically*, by transferring the distance C from A on AB.

2. The Problems 2 and 3 are of very frequent application, for in Geometry we are continually required to draw a line equal to a given line; or to take away from a greater line a part equal to the less; or to lengthen the less so that with the part produced it may equal the greater.

3. The 3rd Problem furnishes the means of constructing a *Scale of Equal Parts;* thus, Take AB of indefinite length towards B, and let C be the given st. line or part that is to be cut off from AB; from AB cut off a part equal to C, as AE; then again from EB another part equal to C, as EF; and so on; the parts in AB are each equal to C and to one another; and AB is a scale of equal parts; for the radii AD, ED′, FD″, GD‴, being each equal to C, AE, EF, FG, and GB are all equal.

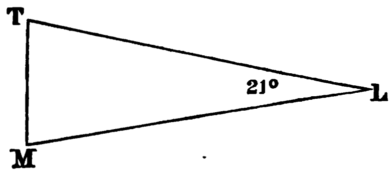




On the same principle we take a line KL, and from one extremity K set off on the line ten equal parts, as in KO; then from O set along the line parts each equal to KO: if the parts between K and O are tenths, the parts 1, 2, 3, 4, 5 will be units; but if the parts between K and O are units, then the parts numbered 1, 2, 3, 4, 5 will be tens, namely, 10, 20, 30, 40, 50. By a scale of this kind, the comparative lengths of lines may be readily measured.

For the advantageous use of a Scale of Equal Parts, we should understand the nature of *Representative Values.* A miniature, of not more than a square inch in surface, is representative of the human face; and a map, on a square of 12 inches, may be representative of a tract in the heavens that takes in distances which we can scarcely conceive. The lines in the miniature and in the map, are in due proportion to those in the face and in the expanse of heaven; and thus they possess a representative value,—they are not the actual distances, but they stand for them. The inch on a scale may indicate a mile, or a thousand miles of distance; but if each portion of a mile, if each mile, or thousand miles, is given of the same relative size, then the map is a true representation; stretch out all its parts in an equal degree, and at last by superposition it would actually cover every point of the wide surface of which it is but the mark or outline.

By such a use of the Scale of Equal Parts, and of the Scale for Angular Magnitude, we can construct figures that are a true index of the positions and real distances of cities, mountains, and seas, and in some respects of the constellations of heaven. For instance,—if by actual observation and measurement it is ascertained that there are two towns each distant thirty-five miles from a third, and that the two, in reference to the third, are at an angle of 21° apart, a plan may easily be drawn which shall correctly shew their situation with respect to each other. From L draw a line L M of 35 from a scale of equal parts; at L with a semicircle make an angle of 21°; and along L T from the same scale set another 35;—the points L, M, T will represent the situations and distances of the three towns.

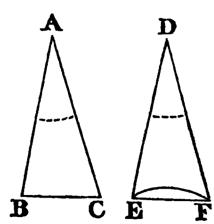


Prop. 4.—Theor.—(*Important.*)

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to each other; they shall likewise have their bases, or third sides, equal; and the two triangles shall be equal, and their angles shall be equal, each to each, viz., those to which the equal sides are opposite.

Demonstration.—Ax. 10. Two straight lines cannot enclose a space.
Ax. 8. Magnitudes which coincide with one another are equal.

Exp.	1	Hyp. 1.	In the △s ABC, DEF, let AB = DE, and AC = DF;
	2	„ 2.	Also let the included ∠ BAC = the included ∠ EDF;
	3	Concl. 1.	then the base BC = the base EF;
	4	„ 2.	and △ABC = △DEF;
	5	„ 3.	also ∠ ABC = ∠ DEF, and ∠ ACB = ∠ DFE.



Dem.	1	Superp.	For, applying △ ABC to △ DEF, so that A is on D, and AB on DE;
	2	D.1, H.1	∵ AB coincides with and is equal to DE,
	3	Ax. 8.	∴ the · B shall coincide with the · E:

DEM.	4	D. 2, 3, H. 2.	Again, $\because$ AB coincides with DE, and $\angle$ BAC = $\angle$ EDF,
	5	Concl.	$\therefore$ the line AC shall fall on the line DF:
	6	H. 1 & Con.	But AC being = DF, the $\cdot$ C shall fall on the $\cdot$ F,
	7	D. 3, 6.	and B falling on E, and C on F, the line BC falls on the line EF;
	8	Con. sup.	For if, though B falls on E, and C on F, BC does not fall on EF, then two st. lines will enclose a space;
	9	Ax. 10.	But this is impossible;
	10	Ax. 8.	therefore the base BC does coincide with and = the base EF:
	11	D. 2, 5, 6, 10	And AB falling on and being equal to DE; AC to DF; and BC to EF,
	12	Ax. 8.	$\therefore$ $\triangle$ ABC coincides with and = $\triangle$ DEF.
	13	Hyp. 1, D. 10	Also, since DE coincides with AB, and EF with BC,
	14	Ax. 8, Concl.	the $\angle$ ABC shall coincide with and equal $\angle$ DEF.
	15	Hyp. 1, D. 7.	And in a similar way $\angle$ ACB = $\angle$ DFE.
	16	Recap.	*Wherefore, if two triangles have two sides, &c.* Q.E.D.

SCHOLIUM.—1. This being the first Theorem in the *Elements*, it is exclusively proved by means of the Axioms.

2. The converse of the 8th Axiom is assumed; namely, that if magnitudes are equal, not merely if they are equivalent, they will also coincide.

3. The equality spoken of in this Proposition, is equality of the sides and of the angles. Triangles may be equal in area, though the sides and angles of the one are not equal to the sides and angles of the other. When the sides and angles mutually coincide, the triangles are named equal triangles; when their areas only are equal, such triangles are called equivalent triangles.

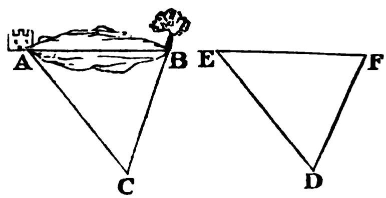
4. Some have taken the 4th Proposition for an axiom. We perceive its truth indeed, almost without demonstration; but the number of axioms in any science should not be needlessly increased; and as this proposition can be naturally established by means of the received axioms of Geometry, it holds its proper place when classed with truths to be demonstrated. It may be more briefly enunciated, thus—"If two triangles have each two sides and their included angle equal, the triangles are equal in every respect."

USE AND APPLICATION.—This Proposition contains the first of the *criteria* by which to infer the equality of triangles, and is applied to various uses: as—

1st. Very frequently in all parts of Geometry to establish the equality of triangles.

***2nd. To ascertain an inaccessible distance,* as AB, the breadth of a lake.**

With an instrument for measuring angles, take the angle at C formed by the lines AC, BC, from the extremities A and B of the inaccessible distance; and with a chain or other measure of length, find the distances CA and CB. The Representative Values of these measurements must now be taken from a Scale of Equal Parts, and drawn on paper, or on



any plane surface: thus, draw a st. line DF of an indefinite length, and at D form an angle, by aid of the graduated semicircle FDE equal to the angle BCA: from a scale of equal parts the distance from A to C is represented in proper proportion by the line DE, and the distance from B to C by DF; consequently, on a principle established in the Sixth Book, and which we now assume as a *Lemma*,—that the sides about similar triangles are proportional,—the line EF will represent in the due proportion the distance from A to B: and if to the same scale we apply the line EF, that distance on the scale will be the representative measurement of the actual distance AB.

N.B.—If the ground near the lake was level enough to admit of setting out the triangle DEF in the actual measurements of CA and CB, we should have EF of the very length of AB, on the principle that two triangles having two sides and their included angle in each equal, have their third sides equal; and if we measure one, as EF, we ascertain the other, AB: but as it is seldom we find the actual surface of a country smooth enough for our purpose, we use the method of Representative Values, or of Geometrical Construction.

Prop. 5.—Theor.

The angles at the base of an isosceles triangle are equal to each other; and if the equal sides be produced, the angles on the other side of the base shall be equal.

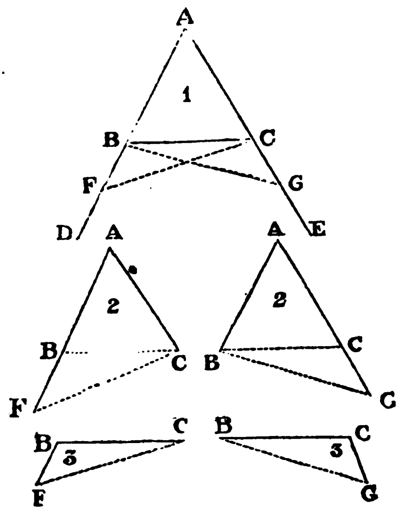
Construction.—Pst. 2. A terminated st. line may be produced to any length in a st. line.

P. 3. From the greater line to cut off a part equal to the less.

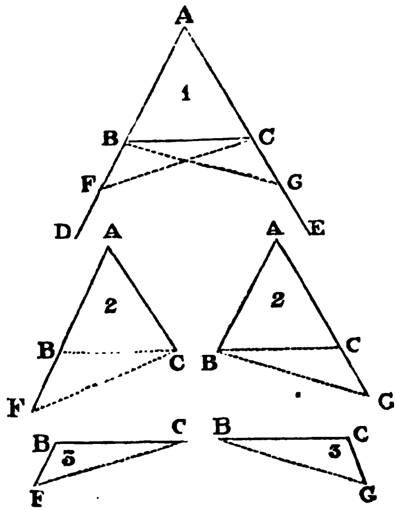
Pst. 1. A st. line may be drawn from any one point to any other point.

Demonstration.—P. 4. If two triangles have two sides and their included angle of one triangle equal to two sides and their included angle of another triangle, the two triangles are equal in every respect.

Ax. 3. If equals are taken from equals, the remainders are equal.



Exp.	1	Hyp. 1.	Let ABC be an isosc. △, having the sides AB and AC equal;
	2	H. 2 & Pst. 2	and let the equal sides be produced indefinitely to D and E;
	3	Concl. 1.	then the ∠s ABC, ACB, at the base are equal;
	4	„ 2.	and the ∠s DBC, ECB, on the other side of the base are equal.
Cons.	1	P. 3.	On AD take any · F, and make AG = EF;
	2	Pst. 1.	join the ·s F and C, G and B, by the |s FC and GB.
Dem.	1	C. 1 & Hyp. 1	∵ AF = AG, AC = AB, and ∠ A is common,
	2	P. 4.	∴ the side FC = GB, and △ AFC = △AGB;
	3	D. 2.	also ∠ ACF = ∠ ABG, and ∠ AFC = ∠ AGB.

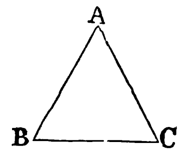


DEM.	4	C.1 & Hyp.1	Again, ∴ AF = AG, and AB = AC,
	5	Sub. Ax. 3.	on taking away the equals AB and AC, the rem. BF = the rem. CG.
	6	D. 5 & 2.	But in △s BCF, BCG, BF = CG, and FC = CB,
	7	D. 3.	and ∠ BFC or AFC = ∠ CGB or AGB;
	8	P. 4.	∴ △ BFC = △ CGB, ∠ FBC = ∠ GCB, and ∠ BCF = ∠ CBG.
	9	D. 3 & 8.	Now, ∠ ABG = ∠ ACF, and the part CBG = the part BCF;
	10	Sub. & Ax. 3.	∴ on taking away ∠ s CBG and BCF, the rem. ∠ ABC = the rem. ∠ ACB,
	11	Remk.	and these are angles at the base.
	12	D. 8.	And ∠ FBC was proved to be equal to ∠ GCB,
	13	Remk.	and these are the angles below, or on the other side of the base.
	14	Recap.	*Wherefore, the angles at the base, &c.*

Q.E.D.

SCHOLIUM.—To assist the learner, the original figure 1 is separated into its parts, 2, 2, and 3, 3; and the equality of the triangles proved by Prop. 4.

Cor.—*Every equilateral triangle is also equiangular.*



Exp.	1	Hyp.	Let ABC be an equil. $\triangle$ AB = BC = AC;
	2	Concl.	then shall its angles be equal $\angle$ A to $\angle$ B to $\angle$ C.
Dem.	1	Def. 24, P. 5	Since AB = AC, $\therefore \angle B = \angle C$,
	2	Hyp. P. 5.	and since CA = CB, $\therefore \angle A = \angle B$;
	3	Ax. 1.	consequently, $\angle C = \angle A$.
	4	Concl.	Hence, the angles are all equal, A to B, B to C, and C to A.
	5	Recap.	*Wherefore, if a $\triangle$ be equilateral,* &c. Q.E.D.

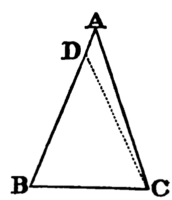
Prop. 6.—Theor.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to the equal angles, shall be equal to one another.

Construction.—P. 3. From the greater line to cut off a part equal to the less.

Pst. 1. A st. line may be drawn from one point to another.

Demonstration.—P. 4. If two triangles have two sides and the included angle equal in each, the triangles are in all respects equal.



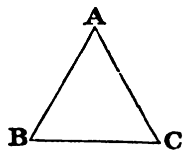
Exp.	1	Hyp.	In $\triangle$ ABC, supposing $\angle B = \angle C$,
	2	Concl.	then the side AB = the side AC.
Sup.	1		For if AB $\neq$ AC, one of them is the greater;
	2		let AB be greater than AC.
Cons.		P.3, & Pst. 1	In BA make BD = AC, and join DC.
Dem.	1	C.	$\because$ DB is made = AC, and BC is common,
	2	Hyp.	and $\because \angle DBC = \angle ACB$;
	3	P. 4.	$\therefore$ DC = AB, and $\triangle$ DBC = $\triangle$ABC.

DEM.	4	*ex abs.*	Thus the less is declared equal to the greater, which is absurd;
	5	Concl.	$\therefore$ AB is not $\neq$ AC, that is, AB = AC.
	6	Recap.	*Wherefore if two angles of a triangle*, &c.

Q.E.D.

COR—*Every equiangular triangle shall also be equilateral.*

EXP.	1	Hyp.	Let $\triangle$ ABC have the $\angle$s equal, A to B, B to C, and C to A;
	2	Concl.	then the sides are all equal, AB to BC, BC to CA, and CA to AB.



DEM.	1	H. & P. 6.	$\because \angle B = \angle C, \therefore AC = AB$.
	2	H. & P. 6.	and $\because \angle B = \angle A, \therefore AC = BC$;
	3	Ax. 1.	and $\therefore AB = BC$.
	4	Concl.	Hence the side AB = BC, BC to CA, and CA to AB.
	5	Recap.	*Wherefore, every equiangular triangle*, &c.

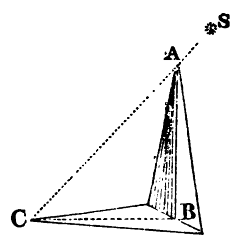
Q.E.D.

SCHOLIUM.—1. This proposition is named the ***converse*** of the 5th. ***Geometrical conversion*** takes place when the hypothesis of the former proposition is made the predicate of the latter, and ***vice versâ***, as in Props. 5 and 6, 18 and 19, 24 and 25 of this Book.

2. Converse theorems are not universally true; for instance, the following ***direct*** proposition is universally true,—"If two triangles have their three sides respectively equal, the three angles of each shall be respectively equal;" but the ***converse*** is not universally true, namely,—"If two triangles have the three angles in each respectively equal, the three sides are respectively equal."—*Potts' Euclid*, p. 48. To the equality of triangles it is always indispensable that one side at least of the one triangle should be given equal to one side of the other triangle.

3. In Geometry there are two modes of Demonstration,—the ***direct***, showing ***why*** a thing is so; and the ***indirect***, proving that it ***must*** be so,—the former being the usual method. ***Direct Demonstration***, as in Prop. 5, is that in which we find intermediate steps which proceed regularly to prove the truth of the proposition: the ***Indirect Method*** is only employed, as in Prop. 6, when the predicate of it admits of an alternative, and one of them ***must*** be true, because they exhaust every case that can possibly exist. We prove that the alternative ***cannot*** be true, and infer, therefore the predicate ***must*** be true. With respect to equality between magnitudes, there are two alternatives—equal, or unequal; and if we prove that inequality cannot or does not exist, of necessity equality must exist.

USE AND APPLICATION.—Hieronymus the historian, records that THALES of Miletus, who was living 546 B.C., measured the height of the pyramids of Egypt, by observing the shadows which they cast when the shadows were as long as the pyramids were high. This would be the case when the altitude of a pyramid, or of any other object, and the perpendicular length of the shadow, were equal.



The height of an object and the length of its shadow are the same, when the light, S, which the object, AB, intercepts, is at an elevation of 45°: this condition being observed, the shadow, BC, is equal to the height, BA; because the angles BCA, BAC, being each half a rt. angle, the sides which subtend them are equal. Thus by measuring the shadow CB, we obtain the height BA.

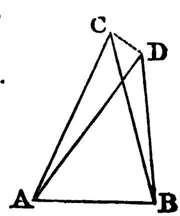
The height would also be obtained by making an observation with a quadrant of altitude; thus,—walk away in a line perpendicular to the altitude of the object, until, at the station C, the quadrant shows A to have an elevation of 45°: the distance gone over from B to C, or BC, will equal BA, the altitude.

PROP. 7.—THEOR.

Upon the same base and upon the same side of it there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those equal which are terminated in the other extremity.

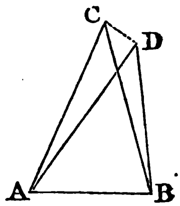
CONST.—Pst. 1. A st. line may be drawn from one point to another.
Pst. 2. A st. line may be produced to any length in a st. line.

DEMONSTRATION.—P. 5. The angles at the base of an isosceles triangle are equal, and if the equal sides be produced, the angles upon the other side of the base are equal.
Ax. 9. The whole is greater than its part.



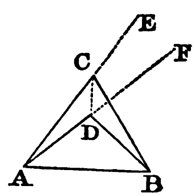
EXP.	1	Hyp. 1.	On AB let there be two △s, ACB, ADB;
	2	" 2.	and let the side CA = the side DA:
	3	Concl.	then it is impossible that CB should equal DB.
SUP.			If possible, let CA = DA, and CB = DB.

CASE I.—*Let the vertices* C *and* D *be without each other.*



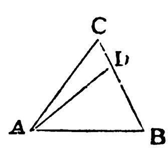
CONST.	1	Pst. 1.	Join C and D.
DEM.	1	Hyp. & P. 5	$\because$ AC = AD, $\angle$ ACD = $\angle$ ADC:
	2	C. & Ax. 9.	But $\angle$ ACD > BCD, $\therefore$ $\angle$ ADC > $\angle$ BCD:
	3	*à fort.*	and much more is $\angle$ BDC > $\angle$ BCD.
	4	H. & P. 5.	Again, $\because$ BC = BD, $\angle$ BDC = $\angle$ BCD;
	5	D. 3.	but $\angle$ BDC is also > $\angle$ BCD;
	6	D. 3, 4.	$\therefore$ $\angle$ BDC is both > and = $\angle$ BCD, which is impossible.

CASE II.—*Let the vertex* D *of* $\triangle$ ADB, *be within the* $\triangle$ ACB.



CONST.	1	Pst. 1 & 2.	Join C to D, & produce AC, AD to E and F.
DEM.	1	Hy. 1 & P. 5	$\because$ AC = AD, $\because$ $\angle$ ECD = $\angle$ FDC.
	2	C. & Ax. 9.	But $\angle$ ECD is greater than $\angle$ BCD; $\therefore$ $\angle$ FDC is greater than $\angle$ BCD;
	3	*à fort.*	and much more is $\angle$ BDC > $\angle$ BCD.
	4	H. & P. 5.	Again, $\because$ BD = BC, $\therefore$ $\angle$ BDC = $\angle$ BCD.
	5	D 3.	But $\angle$ BDC is also > $\angle$ BCD;
	6	D. 4, 3.	$\therefore$ $\angle$ BDC is both > and = $\angle$ BCD, which is impossible.

CASE III.—*When the vertex* D *is on the side* BC, *no demonstration is required.*



7	Recap.	*Therefore, upon the same base, &c.*

Q.E.D.

SCHOLIUM.—The argument made use of in this proposition, is the *Dilemma*, or *double Antecedent*, in which the truth of the one is impossible, if we admit the truth of the other. The argument called the dilemma may, however, have more than two antecedents; and Whately, p. 72, defines the

true dilemma as "*a conditional Syllogism, with several antecedents in the major, and a disjunctive in the minor.*"

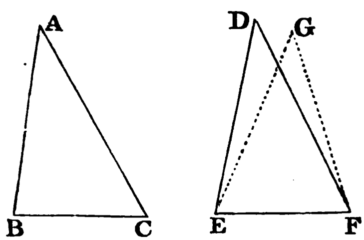
USE.—The only purpose for which this proposition is employed, is to prove Prop. 8.

PROP. 8.—THEOR.—*(Important.)*

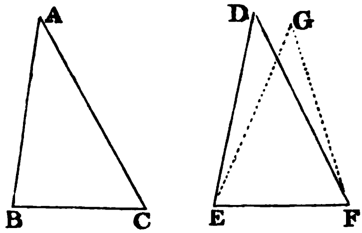
If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one, shall be equal to the angle contained by the two sides equal to them of the other.

DEMONSTRATION.—Pr. 7. On the same side of the same base there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity.

Ax. 8. Magnitudes which coincide are equal to one another.



EXP.	1	Hyp. -	Let the △s ABC, DEF, have AB = DE, AC = DF, and also BC = EF;
	2	Concl.	then ∠ BAC shall equal ∠ EDF.
DEM.	1	Superp.	Apply △ ABC to △ DEF, B on E, and BC on EF.
	2	Hyp.	∵ BC = EF, ∴ C coincides with F.
	3	D.2, Hyp.	Wherefore, ∵ BC coincides with EF, BA and CA shall coincide with ED, FD.
	4	Supp. 1.	For, suppose that BC coincides with EF,
		" 2.	but BA and CA not with ED, FD, but with other lines EG, FG,

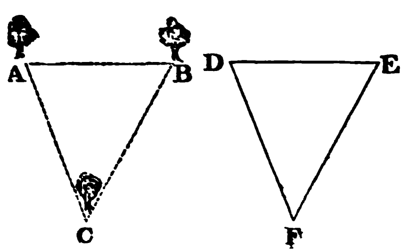


DEM.	5	Concl.	then on this supp., in $\triangle$s E D F, E G F, on the same side of E F, E D shall = E G, and also F D = F G;
	6	P. 7.	which is impossible.
	7	D. 2, 3.	$\therefore$ since B C coincides with E F, the sides B A, C A, coincide with E D, F D:
	8	D. 7.	Wherefore, $\angle$ B A C must coincide with $\angle$ E D F;
	9	Ax. 8.	and $\therefore$ $\angle$ B A C is = $\angle$ E D F.
	10	Recap.	*Therefore, if two triangles have two sides,* &c. Q.E.D.

SCHOLIUM.—The Equality established is that of the angles,—but the sides being equal, the triangles also must be equal. This is the second criterion of the equality of triangles.

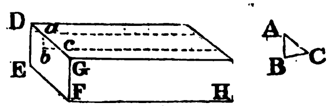
USE.—1. By the aid of this proposition, and of Prop. 22, *the angle at a given point* C, *made by lines from two objects, as* A *and* B, *may be determined without a theodolite.*

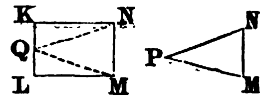
Measure the distances A B, B C, C A,—and from a scale of equal parts construct a triangle D E F, the sides of which, D E, E F, and F D, will be representative of the distances A B, B C, and C A: then with the semicircle find the number of degrees in angle F; and as the triangles A C B, D E F are similar, that number of degrees will also be the measure of angle C.



2. When the instruments for angular magnitude cannot be employed, by reason of the inequalities of surface, or the difficulty of placing the instruments, this proposition is useful for measuring and cutting angles in a solid body, as in a block of stone, or for bevelling, *i. e.*, for giving the desired

shape to the angular edges of timber, &c. For instance, a groove of the same triangular shape and size with the triangle ABC, is to be cut in a block of marble DEFGH. At the point in the edge DG of the block, where the groove is to commence, set off a line ac equal to AC; and on the plane surface DEFG, with ac for one side, construct a triangle abc, with sides equal to the side ABC: abc will be the end of the groove, and if the guidance of abc be followed, the whole groove when finished will be of the same angular magnitude with ABC.





On the same principles, a beam of timber, the end of which is represented by KLMN, may be bevelled so that the bevelled edge shall be of the same angle with a given angle NPM; for, on the end of the beam draw a triangle, the sides of which shall equal those of the triangle NMP; then by Prop. 8, the bevelled edge NQM is equal to the given angle NPM.

Prop. 9.—Prob.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.

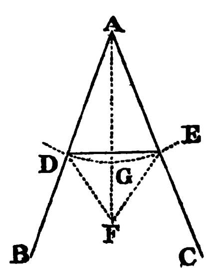
Solution.—P. 3. From the greater line to cut off a part equal to the less.

P. 1. On a line to draw an equil. triangle.

Pst. 1. A line may be drawn from one point to another.

Demonstration.—P. 8. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one, shall be equal to the angle contained by the two sides equal to them of the other.

Exp.	1	Datum.	Let the given ∠ be BAC;
	2	Quæs.	it is required to bisect it.
Cons.	1	Assum.	Take any point D in AB;
	2	P.3 & Pst.1	on AC make AE = AD, and join DE,
	3	P.1 & Pst.1	and on DE construct an equil. △, DFE, and join AF;
	4	Sol.	then ∠ BAF shall = ∠ CAF, the ∠ BAC being bisected.



DEM.	1	C. 2 & P. 1.	∵ AD = AE, DF = EF, and AF common,
	2	P. 8.	∴ ∠DAF is = ∠EAF.
	3	Recap.	*Wherefore, the* ∠BAC *is bisected by* AF.

Q.E.F.

SCHOLIUM.—1. The bisection of the arc which measures an angle, is also effected by the bisection of the angle. The arc DGF is the measure of the ∠BAC, or DAE; the ∠DAG is one-half, and EAG the other half, of ∠ DAE; and halves of the same being equal, the arc DG is equal to the arc GE.

2. An isosceles triangle would serve equally well for the solution and demonstration.

3. By successive bisections an angle may be divided into any number of equal parts indicated by a power of two, as into four, eight, sixteen, thirty-two, &c., equal parts.

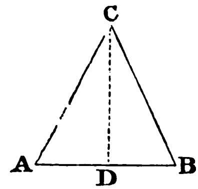
4. Hitherto no method has been discovered of geometrically *trisecting an angle*, so that the division of the quadrant of 90° into single degrees, is in part effected mechanically: by simple bisection, we divide 90° into two 45°; by setting off a semi-diameter from one extremity of the quadrant, we cut off an arc of 60°; 60° bisected, gives 30°; and 30° bisected gives 15°: but for the division of the 15° we require to have the means of trisecting an angle, which means Geometry does not supply. But having mechanically divided an arc of 15° into three 5°, and from one arc of 5° set off an arc of 3°, the simple bisection of the remaining arc of 2° gives an arc of 1°,—a unit in the measure of the circumference. Of course this is only a practical, not a theoretical proof.

If, however, instead of taking an arbitrary quantity, 360, as the measure of the equal parts in the circumference of a circle, those who first made such division had followed the strictly geometrical process of this 9th Proposition, they would have arrived at a unit for the degrees in a given circumference, with as much absolute certainty as they do now at the unit for a scale of equal parts. By making use of the powers of 2, and by their aid dividing the circle, the unit of the division is *demonstrably* accurate. Suppose the number of equal parts into which the circle had been divided, had been represented by the 9th power of 2, or 512, the bisection would have given 256 for the semicircle; 128 for the quadrant; and 64 for the octant: and 64, by successive bisection, would have given 32, 16, 8, 4, 2 and 1 equal parts. Thus every step in the division would have been strictly in accordance with geometrical verities. Again—the unit of such degrees represented by 64 equal parts, would in the same way have been divisible into 32, 16, 8, 4, 2, and 1 minutes, and so on, to whatever extent of minuteness we might wish to carry our bisections.

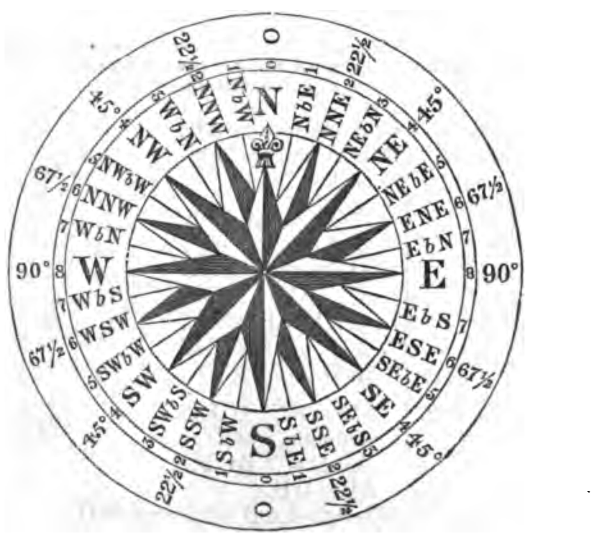
Probably no fact in geometrical measurements more clearly shows the unscientific nature of the early geometry, than the division of the circle into 360 equal parts. It is now, however, too late to attempt an alteration on purely geometrical grounds; and, fortunately, there is no real inconvenience or inaccuracy in the received method;—for an arc of the 360th part of a circle, is in practice as readily obtained as the arc of the 512th part of the same circle would be. We have only to bear in mind that Plane Geometry does not supply the means for any division of a circle, except by the method of bisections, *i. e.*, by using in regular series the powers of 2.

USE AND APPLICATION.—1. Practically the angle BAC, in the figure to P. 9, would be bisected by drawing the arc DGE, and with any radius from D and E drawing arcs intersecting in F; AF is the bisecting line.

2. By Prop. 9, we show that the angles at the base of an isosceles triangle are equal; for bisecting ∠ ACB by CD, CA is equal to CB, CD common, and ∠ ACD equal ∠ BCD; therefore by P. 4, ∠ CAD equals ∠ CBD.



3. Also that the line which bisects the vertical angle of an isosceles triangle, bisects the base perpendicularly; for AC equalling BC, DC being common, and ∠ ACD by construction equalling ∠ BCD, by Prop. 4, AD equals DB, and △ADC equals △ BDC, and ∠ ADC equals ∠ BDC; consequently, by Def. 10, DC is perpendicular to AB.



4. The *Mariner's Compass* is divided into its 32 parts or points by Props. 9 and 10. The bisection of the diameter by another diameter at rt. angles, gives the cardinal points N., E., S., W.; the quadrants, being bisected, give the points N.E., S.E., S.W., N.W.; and these bisections are continued until the number of equal divisions of the circle amounts to 32,—the arc at each pair of points enclosing an angle of 11° 15′. This instrument is used for taking the bearings or directions of places from some central station of observation; and the correctness of the method depends on the physical law

that a magnetized needle or index, working freely on a pivot, always, from the same place, points in the same direction. To be accurate in making the observation, it is requisite to know the amount of deviation from the true North, at any given place. For Great Britain, the deviation amounts to about 24° west of north; so that when the magnetized steel index, or needle, points 24° west, the pointer on the compass card marked N., indicates the true North.

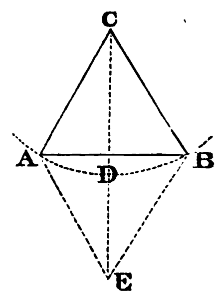
Prop. 10—Prob.

To bisect a given finite straight line.

Solution.—P. 1. On a given line to construct an equilateral triangle. P. 9. To bisect a given rectilineal angle.

Demonstration.—P. 4. Two triangles are equal in every respect when two sides and the included angle of one are equal to two sides and the included angle of the other triangle.

Exp.	1	Datum.	Given the st.line AB;
	2	Quæs.	it is required to bisect it.
Cons.	1	P. 1.	On AB make an equil. △ ABC;
	2	P. 9.	let CD make ∠ ACD = ∠ BCD;
	3	Sol.	then AD = BD, *i. e.*, AB is bisected in D.
Dem.	1	C. 1 & 2.	∵ AC = BC, CD is common, and ∠ ACD = ∠ BCD.
	2	P. 4.	∴ AD = DB.
	3	Recap.	*Wherefore* AB *is bisected in* D. Q.E.F.



Scholium.—By successive bisections a line may thus be divided into any number of equal parts indicated by a power of 2; as, 4, 8, 16, 32, 64, &c.

Use.—1. In practice, the line AB would be divided into equal parts by drawing with equal radii arcs intersecting in C and E; the line CE will bisect AB.

2. Also, if the length of the line AB be ascertained by means of a scale of equal parts, the division into any required number of equal parts, as 2,

3, 4, &c., will be effected by dividing the numerical value by 2, 3, 4, &c., as the case may be.

3. But for a ready and infallible method of bisecting a line, Prop. 10 cannot be dispensed with.

Prop. 11.—Prob.

To draw a st. line at right angles to a given straight line from a given point in the same.

Solution.—P. 3. From the greater of two lines to cut off a part equal to the less.

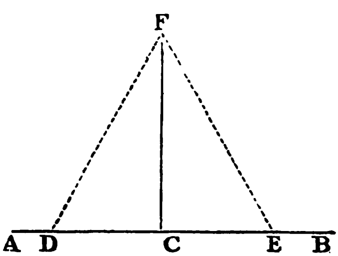
P. 1. On a given line to construct an equilateral triangle.

Pst. 1. Two points may be joined by a line.

Demonstration.—P. 8. In two triangles, if two sides and the base of one triangle be equal to the two sides and the base of another triangle, the angle between the two equal sides of the one is equal to the angle between the two equal sides of the other.

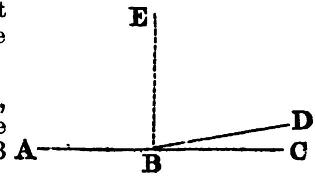
Def. 10. When a straight line on another st. line makes the adjacent angles equal, each of the angles is a right angle.

Ax. 1. Things equal to the same thing are equal to one another.



Exp.	1	Data.	Given the st. line AB, and C a point in it;
	2	Quæs.	to draw from C a st. line at ⊥ AB.
Cons.	1	P. 3.	In A C take D, and make CE = CD;
	2	P. 1 & Pst. 1	on DE construct the equil. △ FDE, and join FC;
	3	Sol.	then ∠ DCF and ECF are equal, and consequently rt. angles.
Dem.	1	C. 1 & 2.	∵ DC = EC, FC common, and DF = EF,
	2	P. 8.	∴ ∠ DCF = ∠ ECF:
	3	C. 2.	And by construction they are adjacent angles;
	4	Def. 10.	∵ the ∠ s DCF and ECF are rt. angles.
	5	Recap.	*Wherefore from the given point* C, &c. Q.E.F.

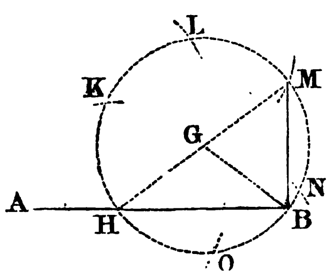
Cor.—*Hence it may be shown that two st. lines cannot have a common segment.*



Exp.	1		If possible, let the segment AB be common to the lines ABC and ABD.
Cons.	1	P. 11.	At B draw BE, making the ∠s with AB rt. angles.
Dem.	1	Hyp.	∵ AB and BC make one st. line ABC, and ∠ABE is a rt. angle;
	2	Def. 10.	∴ ∠ABE = ∠EBC:
	3	Hyp.	Again, ∵ AB and BD form one line ABD, and ∠ABE is a rt. angle;
	4	Def. 10.	∵ ∠ABE = ∠EBD:
	5	D.2,4,Ax.1	Wherefore ∠EBD = ∠EBC; which is impossible.
	6	Recap.	*Therefore two st. lines cannot have a common segment.* Q.E.D.

Scholium.—1. This corollary should be ranked among the axioms; for it is assumed in Prop. 4, where it is taken for granted that if certain lines, placed on one another, coincide for any portion of their length, they must coincide throughout. It is also assumed in Prop. 8.

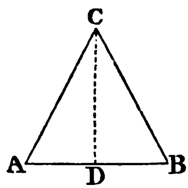
2. When the given point is at the extremity of a given line, the line from that extremity should be produced, and the rt. angle be then constructed:



3. Or the following method may be adopted:—Let AB be the given line, and B the extremity from which the perpendicular is to be raised. Take any point G above the line AB, and with radius GB describe a circle cutting AB in H; join HG and produce it to M; or from H set the same radius GB on HK, KL, and LM; M is the point from which if a line be drawn to B, that line MB will be perpendicular to AB. The radius equals the chord of 60°; three times 60° equal 180°; and 180° is the semicircle: and by P. 31, bk. iii., the angle HBM in a semicircle is a rt. angle.

USE AND APPLICATION.—1. By this proposition *the Square* is constructed, an instrument employed for ascertaining the perpendicular to a horizontal line, and for all purposes for which right angles are needed.

2. *On a given line* AB, *to describe an isosceles triangle of which the perpendicular height* CD, *is equal to the base* AB.



CONS.	1	P. 10, 11, & 3	Bisect AB in D, and make DC perpen. and equal to AB;
	2	Pst. 1.	join AC and BC;
	3	Sol.	the figure ABC is the isosc. △ required.
DEM.	1	C. 1.	∵ AD=DB, DC is common, and ∠ ADC = ∠ BDC;
	2	P. 4.	∴ AC = BC.
	3	C. 1.	And the perp. DC has been made equal to AB. Q. E. F.

—

PROP. 12.—PROB.

To draw a perpendicular to a given st. line of unlimited length from a given point without it.

SOLUTION.—Pst. 3. A circle may be drawn from any centre at any distance from that centre.

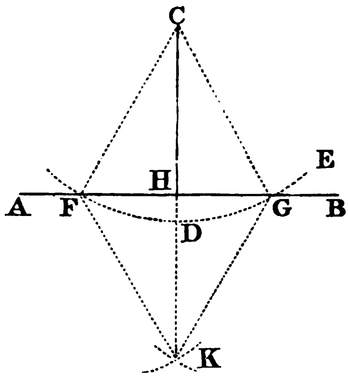
P. 10. To bisect a given st. line.

Pst. 1. A line may be drawn from one point to another.

DEMONSTRATION.—Def. 15. The radii of the same circle are all equal.

P. 8. If two triangles have two sides in one equal to two sides in the other, and the base equal to the base, the angles contained by the two pair of equal sides are equal.

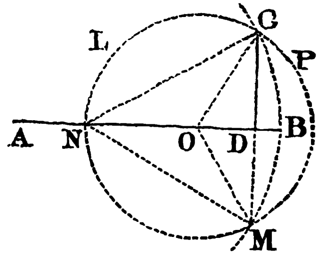
Def. 10. A st. line at rt. angles to another st. line, is perpendicular to it.



Exp.	1	Data.	Given the st. line AB, and the · C out of it;
	2	Quæs.	required from C a perpendicular to AB.
Cons.	1	Assum.	Take D on the other side of AB;
	2	Pst. 3.	from C with rad. CD draw the arc EGF, cutting AB in F and G;
	3	P.10, Pst.1	bisect FG in H, and join C and H, C and F, and C and G;
	4	Sol.	then CH is perpendicular to AB.
Dem.	1	C.3, Def.15	∵ FH = HG, CF = CG, and HC common,
	2	P. 8.	∴ ∠CHF = ∠CHG:
	3	C. Remk.	Now these are adjacent angles;
	4	Def. 10.	∴ CH is perpendicular to AB.
	5	Recap.	*Therefore from the given point*, &c. Q.E.F.

Scholium.—1. The properties of the circle form the subject of the third book, but in the construction for the 12th Prop., the Lemma is borrowed from bk. iii., that the circle will intersect the line in two points.

2. If the given point C is over the extremity of the given line AB;—In AB take any point O, and with rad. OC describe the arc CLM; and from N another point in AB, with rad. NC, the arc CPM: join the points of intersection C and M, CD is the perpendicular required.



In the triangles NCO, NMO,—the three sides of the one are equal to the three sides of the other, and by P. 8, the angle CNO equals the angle MNO: thus in the triangles NDC, NMD, two sides and the included angle of one are equal to two sides and the included angle of the other:—therefore by P. 4, ang. ADC equals ang. ADM, and they are adjacent angles; hence CD is perpendicular to AB.

Use and Application.—1. In practice, the problem will be solved by drawing, as in the figure to P. 12, the arc FDE, and from the points F and G, with equal radii, describing arcs intersecting in K; by joining CK, the perpendicular to AB will be drawn.

2. This problem is indispensable to all Artificers, Surveyors, and Engineers.

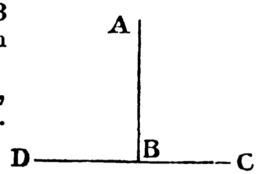
Prop. 13—Theor.

The angles which one st. line makes with another upon one side of it are either right angles, or together equal to two right angles.

Construction.—P. 11. To draw a st. line at right angles to a given st. line from a given point in the same.

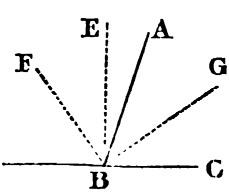
Demonstration.—Def. 10. When a line standing on another line makes the adjacent angles equal, each of the angles is a right angle.
Ax. 8. Magnitudes which exactly fill the same space are equal.
Ax. 2. If equals be added to equals, the sums will be equal.
Ax. 1. Magnitudes which are equal to the same, are equal to each other.

Exp.	1	Hyp.	Let the st. line AB make angles with the st. line DC,
	2	Concl. 1.	then the ∠s CBA, ABD, are two rt. angles,
	3	„ 2.	or are together = two rt. angles.



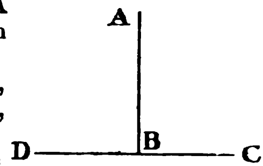
Case I.—*Suppose that the ang.* CBA *is equal to ang.* ABD;

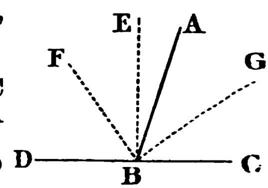
		Def. 10.	then each of the angles is a right angle.



CASE II.—*But suppose that the angle* CBA *is not equal to angle* ABD;

CONS.	1	P. 11.	At B in DC draw BE at rt. angles to CD,
	2	Def. 10.	then ∠s CBE, EBD are two rt. angles.
DEM.	1	Ax. 8, Add.	∵ ∠ CBE = ∠s CBA + ABE, to each add EBD,
	2	Ax. 2.	∴ ∠s CBE, EBD, = ∠s CBA, ABE, and EBD.
	3	Ax. 8, Add.	Again, ∵ ∠ DBA = ∠s DBE & EBA, to both add ∠ ABC;
	4	Ax. 2.	∴ ∠s DBA + ABC = ∠s DBE, EBA and ABC.
	5	D. 2.	But ∠s CBE, EBD = ∠s DBE, EBA and ABC;
	6	Ax. 1.	∴ ∠s CBE, EBD = ∠s DBA, ABC;
	7	C.	Now ∠s CBE, EBD are two rt. angles,
	8	Ax. 1.	∴ ∠s DBA, ABC together = two rt. angles.
	9	Recap.	*Wherefore, the angles which one line, &c.*





Q.E.D.

SCHOLIUM.—1. A rt. angle FBG is formed by bisecting the angles ABD, ABC.

2. If one angle be a rt. angle, the other is a rt. angle; if one be obtuse, the other is acute: and if one be acute, the other is obtuse.

3. A semicircle is the measure of two right angles; and all the angles formed by any number of lines converging to one point, on one side of another line, are together equal to two right angles.

4. The *supplement* to an angle is what it is deficient of two rt. angles; thus, ABD is the supplement of angle ABC: the *complement*, what is wanting to make up one rt. angle; as, ABE is the complement of ang. ABC.

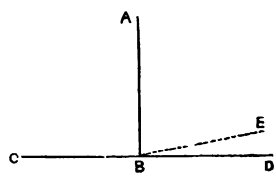
USE AND APPLICATION.—This theorem is of frequent use in Trigonometry and Astronomy. When we know one of the angles which a st. line *meeting* another st. line makes, at a point in it, we in fact know the other, for the two angles are always equal to 180°; and if we subtract the given arc from 180° we have the other angle: thus, let ang. ABC equal 70°, ABD equals 180—70, or 110.

PROP. 14.—THEOR.

If at a point in a st. line, two other lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two lines shall be in one and the same line.

CONSTRUCTION.—Pst. 2. A straight line may be produced to any length in a st. line.

DEMONSTRATION.—P. 13. The angles made by one st. line with another on the same side of it, are equal to two rt. angles,
Ax. 1. Things equal to the same thing are equal to one another.
Ax. 3. If equals be taken from equals, the wholes are equal.



EXP.	1	Hyp.	At B in AB, let BC and BD make the adjacent $\angle$s ABC, ABD = two rt. angles;
	2	Concl.	then BC and BD will form one and the same st. line CD.
CONS.	1	Supp.	Should BD not be a production of the st. line CB,
	2	Pst. 2.	make BE a continuation of CB.
DEM.	1	Hyp.	$\because$ AB with the st. line CBE makes the $\angle$s ABC, ABE,
	2	P. 13.	$\therefore$ these $\angle$s ABC, ABE are = two rt. angles:
	3	Hyp.	But $\angle$s ABC and ABD also = two rt. angles,
	4	Ax. 1.	$\therefore$ $\angle$s ABC and ABE = $\angle$s ABC and ABD:
	5	Sub.	taking away the common $\angle$ ABC,
	6	Ax. 3.	then the rem. $\angle$ ABE = the rem. $\angle$ ABD,
	7	*ex abs.*	and the less $\angle$ ABE = the greater $\angle$ ABD;
	8	Concl.	$\therefore$ BE is not in the same st. line with CB.
	9	Sim.	In the same way no line except BD is a continuation of CB;
	10	Concl.	$\therefore$ BD is in one and the same st. line with CB.
	11	Recap.	*Wherefore if at a point in a line, &c.* Q.E.D.

SCHOLIUM.—1. The words "upon the opposite sides of it" are of essential importance; for two lines may make with a third, two angles equal to two rt. angles, and yet the two lines not be in one st. line.

2. The fourteenth proposition is the converse of the thirteenth.

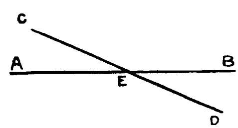
PROP. 15.—THEOR.

If two st. lines cut one another, the opposite or vertical angles shall be equal.

DEMONSTRATION.—P. 13. The angles which one st. line makes with another st. line upon one side of it, are either two rt. angles, or equal to two rt. angles.

Ax. 1. Things equal to the same are equal to each other.

Ax. 3. If equals be taken from equals, the remainders are equal.



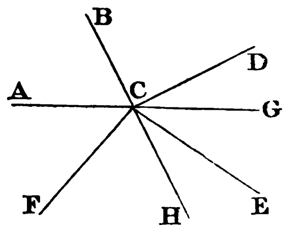
EXP.	1	Hyp.	Let the st. lines AB, CD intersect in the point E;
	2	Concl.	then $\angle$ AEC = $\angle$ DEB, & $\angle$ CEB = $\angle$ AED.
DEM.	1	Hyp.	∵ AE makes with CD the $\angle$ s CEA and AED,
	2	P. 13.	∴ these $\angle$ s CEA and AED = two rt. angles.
	3	Hyp.	And ∵ DE makes with AB the $\angle$ s AED, DEB;
	4	P. 13.	∴ these $\angle$ s AED and DEB = two rt. angles;
	5	Ax. 1.	∴ also $\angle$ s CEA, AED = $\angle$ s AED, DEB:
	6	Sub.	Take away the common angle AED,
	7	Ax. 3.	and $\angle$ CEA will = $\angle$ DEB.
	8	Sim.	In like manner ang. CEB is proved equal to ang. AED.
	9	Recap.	*Therefore, if two st. lines cut one another,* &c.

Q.E.D.

COR. 1.—*The angles formed by two lines,* AB *and* CD, *cutting each other in one point* E, *are together equal to four right angles.*

DEM.	1	P. 13.	∵ $\angle$ s AEC and CEB = two rt. angles,
	2	Pr. 13.	and $\angle$ s AED and DEB also = two rt. angles;
	3	Add.	by adding the equals to the equals,
	4	Ax. 2.	∴ the $\angle$ s AEC, CEB, AED, and DEB = four rt. angles.

COR. 2.—*And all the angles formed by any number of lines,* AC, BC, DC, EC, FC, *&c., diverging from a common centre,* C, *are together equal to four rt. angles.*



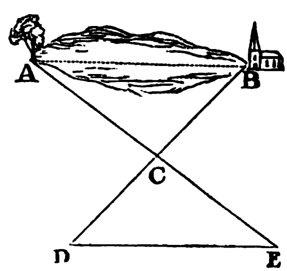
CONS.	1	Pst. 1.	Produce any two of the given lines, as AC to G, BC to H.
DEM.	1	Cor.1,15	∵ AG and BH intersect in the point C, the ∠s ACB, BCG, ACH and HCG = four rt. ∠s.
	2	P. 13.	But ∠s ACB, BCD, DCG = ∠s ACB and BCG;
	3	P. 13.	also ∠s ACF, FCE, and ECG = ∠s ACH and HCG;
	4	Concl.	∴ all the angles diverging from C are together equal to four rt. angles.

SCHOLIUM.—1. This proposition might be briefly proved by saying that the opposite angles are equal, because they have the same supplement.

2. "This Prop. is the developement of the definition of an angle. If the lines at the angular point be produced, the produced lines have the same inclination to one another as the original lines have."—*Potts' Euclid,* p. 49.

3. The converse of this proposition is, "If four lines meeting in a point make the vertical angles equal, each alternate pair of lines shall be in one and the same st. line."

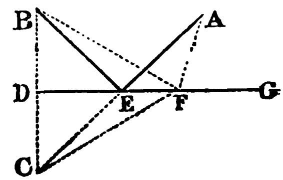
USE AND APPLICATION.—1. By an easy application of Prop. 15, we find *the breadth of a lake,* or the distance between two inaccessible objects, A and B.



From any station C, accessible both to A and B, measure the lines CA and CB, and produce them until CE is equal to CA, and CD to CB; join D and E, the distance DE will equal the distance AB.

For the lines CA, CB equal the lines CE, CD; and the angle ACB equals the angle DCE; therefore by Prop. 4, the lines AB and DE are equal.

2. On the principle that the angle of reflection equals the angle of incidence, this proposition is useful to the billiard player to enable him with a ball A to strike by reflection another ball B.



Let DG be one side of the table; imagine a line from B to be perpendicular to DG, and produced until DC equals BD: if now the ball A be driven in the line AEC, so that if it was not for the side DG it would reach C, the ball A on striking E will be reflected from its course so as to reach the ball B; *i. e.*, the angle of reflection BED will be equal to the angle of incidence AEG.

In triangles BDE, CDE, the side BD is equal to DC, DE common, and the included angles at D equal: therefore by Prop. 4 the ang. BED is equal to the ang. CED; but by Prop. 15, ang. CED is equal to ang. AEG: consequently the angle BED equals the angle AEG.

3. Another use of this proposition is, *to determine the number and kind of polygons which may be joined to cover a given space.*

The circle, consisting of 360°, is the measure for the sum of all the angles that can possibly be drawn from a point; and no regular right-lined figures can fill up the space around a point, unless the angle contained by any two of the sides of the figure be a measure of 360°. The only regular right-lined figures of which the angles contained by two sides are measures of a circle, are the equilateral triangle, the square, and the hexagon; for 60°, 90°, and 120°, the respective angles of these figures, are measures of 360°. The angle formed by two conterminous sides of the pentagon contains 108°, of the heptagon $128\frac{4}{7}$°, of the octagon 135°, &c.; but none of these numbers are measures of 360°.

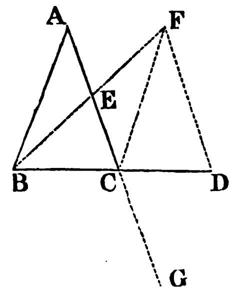
Prop. 16.—Theor.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Construction.—P. 10. To bisect a given st. line.
P. 3. From the greater line to take a part equal to the less.
Pst. 1. A st. line may be drawn from one point to another.
Pst. 2. A st. line may be produced to any length in a st. line.

Demonstration.—P. 15. If two st. lines cut one another, the vertical angles shall be equal.
P. 4. If two triangles have in each two sides and the included angle equal, they are equal in all respects.
Ax. 9. The whole is greater than its part.

Exp.	1	Hyp.	Let BC, a side of △ ABC, be produced to D;
	2	Concl.	then the ext. ∠ ACD > ∠ CBA, or than ∠ CAB.
Cons.	1	P. 10 & Pst. 1	Make CE = AE, and join BE;
	2	Pst. 2, P. 3, Pst. 1.	produce BE so that BE = EF, and join FC.
Dem.	1	C. 1 & 2.	∵ AE = EC, and BE = EF,
	2	P. 15.	and ∠ AEB = ∠ CEF;
	3	P. 4.	∴ △AEB = △CEF, and ∠ BAE = ∠ FCE.
	4	C.	But ∠ ECD or ACD > ∠ ECF.
	5	Ax. 9.	and ∴ ∠ ACD > ∠ BAC.
	6	Sim. P. 10.	In like manner produce AC to G, and bisect BC,
	7	D. 1—5.	and ∠ BCG, *i.e.*, ∠ ACD, will be shewn to be > ∠ ABC.
	8	Recap.	*Therefore if one side of a triangle be produced,* &c. Q.E.D.

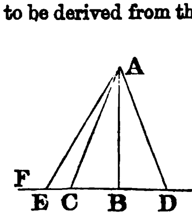


Scholium.—1. We may illustrate Prop. 16 in the following way:—Let triangle ABC slide along the st. line BD until the point B covers the point C; it is obvious that vertex A, or rather F, will be at the right hand of the point A, and that the line CF must be within the ang. ACD; the ang. FCD or ABC will therefore be less than the ext. ang. ACD.

2. Each angle of a triangle is less than the supplement of either of the other angles.

Use and Application.—Among the conclusions to be derived from this proposition are—

1. *Only one perpendicular* AB *can be drawn from a point* A *to a st. line* FD.



Let AB be perp. to BC, another line, as AC, is not perpendicular. For ext. ang. ABD by P. 16, is greater than ACB, and ABC also being a rt. angle and greater than ACB, ACB is not a rt. angle, nor is AC perp. to FD.

2. If a line AC make the angle ACB acute, and ACF obtuse, the perpendicular AB from A, shall fall on the same side as the acute angle does; for if we suppose AE, the line on the side of the obtuse angle to be perpendicular, and AEF to be a rt. angle, then the right angle AEF would be greater than the obtuse angle ACE.

3. In measuring triangles, parallelograms, and trapeziums, and in reducing them to rectangular figures, these and similar conclusions are of great use.

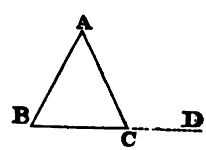
—

Prop. 17.—Theor.

Any two angles of a triangle are together less than two right angles.

Construction.—Pst. 2. A st. line may be produced to any length in a st. line.

Demonstration.—P. 16. If one side of a triangle be produced, the ext. angle is greater than either of the int. opposite angles.
Ax. 4. If equals be added to unequals, the wholes are unequal.
P. 13. The angles formed by one line on another, are together equal to two rt. angles.



Exp.	1	Hyp.	Let ABC be a triangle;
	2	Concl.	then the $\angle$ s A and B are $<$ two rt. angles; also $\angle$ s B and C, and C and A.
Cons.	1	Pst. 2.	Produce BC to D.
Dem.	1	Const.	$\because$ $\angle$ ACD is the exterior angle,
	2	P. 16.	$\therefore$ $\angle$ ACD $>$ $\angle$ B or $\angle$ A.
	3	Add.	To each of these unequals add the $\angle$ ACB.
	4	Ax. 4.	Then $\angle$ s ACD and ACB $>$ $\angle$ s ABC and ACB.
	5	P. 13.	But $\angle$ s ACD and ACB $=$ two rt. angles.
	6	D. 4.	$\therefore$ $\angle$ s ABC, ACB $<$ two rt. angles.
	7	Sim.	In like manner $\angle$ s BAC, ACB $<$ two rt. angles.
	8	Sim.	and $\angle$ s BAC, ABC $<$ two rt. angles.
	9	Recap.	*Therefore, any two angles of a triangle*, &c.

Q.E.D.

Scholium.—1. This Proposition is explanatory of the twelfth Axiom, and the converse of it.

2. Both the sixteenth and the seventeenth propositions will be included in the thirty-second, in which it will be proved that the three angles of a triangle together equal two right angles.

3. The seventeenth Prop. is useful for demonstrating some of those that follow.

—

PROP. 18.—THEOR.

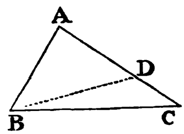
The greater side of every triangle is opposite to the greater angle.

CONSTRUCTION.—P. 3. From the greater line to take a part equal to the less.

Pst. 1. A st. line may be drawn from one point to another.

DEMONSTRATION.—P. 5. The angles at the base of an isosceles triangle are equal.

P. 16. If one side of a triangle be produced, the ext. angle is greater than either o fthe int. and opposite angles.



EXP.	1	Hyp.	In △ ABC, let AC be greater than AB;
	2	Concl.	then ∠ ABC is > ∠ ACB.
CONS.	1	P.3 & Pst.1	Make AD = AB, and join BD.
DEM.	1	C. & P. 5.	∵ AD = AB, ∴ ∠ ADB = ∠ ABD:
	2	P. 16.	But ext. ∠ ADB > int. ∠ DCB,
	3	D. 1.	and ∠ ADB = ∠ ABD;
	4	D. 2, 3.	∴ ∠ ABD > ∠ DCB;
	5	*à fort.*	much more is ∠ ABC > ∠ ACB.
	6	Recap.	*Therefore, the greater side of every triangle,* &c. Q.E.D.

SCHOLIUM.—The argument on which the conclusion depends is named "*à fortiori,*" *by the stronger reason,* and proves that a given predicate belongs in a greater degree to one subject than to another; as in the Syllogism,—Y is greater than Z, and X greater than Y; much more is X greater than Z.

USE AND APPLICATION.—For the demonstration of other propositions, as Prop. 19.

—

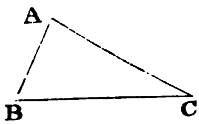
PROP. 19.—THEOR.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

DEMONSTRATION.—P. 5. The angles at the base of an isosceles triangle are equal.

P. 18. The greater side of every triangle is opposite to the greater angle.

Exp.	1	Hyp.	In $\triangle$ ABC let $\angle$ABC $>$ $\angle$ACB;
	2	Concl.	then side AC is $>$ side AB.



Sup.—*The side* AC *must be greater than, equal to, or less than* AB.

Dem.	1	Sup. & P.5.	If AC = AB, then $\angle$ABC = $\angle$ACB:
	2	Hyp.	but $\angle$ABC $\neq$ $\angle$ACB;
	3	D. 1 & 2.	$\therefore$ AC $\neq$ AB.
	4	Sup.&P.18	Again, if AC is $<$ AB, then $\angle$ABC is $<$ $\angle$ACB:
	5	Hyp.	but $\angle$ABC is $\nless$ $\angle$ACB;
	6	D. 4 & 5.	$\therefore$ AC is $\nless$ AB.
	7	D. 3 & 6.	Now, AC is neither equal to, nor less than, AB;
	8	Concl.	$\therefore$ AC is greater than AB.
	9	Recap.	*Wherefore, the greater angle of every triangle,* &c. Q.E.D.

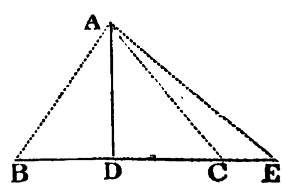
Scholium.—1. This proposition is the converse of the eighteenth, and bears the same relation to it as the 6th does to the 5th Prop.

2. Propositions 5, 6, 18, and 19, may be combined into one proposition, thus,—"One angle of a triangle is greater than, equal to, or less than, another angle, as the side opposed to the one is greater than, equal to, or less than, the side opposed to the other; and *vice versâ.*"

3. By aid of Props. 17, 18, and 19, we may prove that from the same point there can be drawn but one perpendicular to a given line, and that this perpendicular is the shortest of all the lines from the given point to the given line.

4. As from a given point only three lines can be drawn perpendicular to each other, it is impossible to imagine that there are more than three species of quantity,—a Line, a Surface, and a Solid.

Use and Application.—1. *The perpendicular* AD, *is the shortest line from a point* A *to a given line* BC.



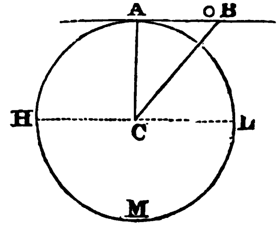
Because ang. ADB is a rt. ang., the ang. formed by any other line from A, as ABC, is acute, by P. 17; and by P. 19 the side AD is less than AB: and in the same way we can prove that no other line is less than AD: therefore the perpendicular, is the shortest line from a point to a given st. line.

For this reason the Perpendicular is made use of in measuring: and irregular figures are reduced to those of which the angles are rt. angles.

2. *From the same point* A *only two equal lines,* AB *and* AC, *can be drawn to a given line* BE.

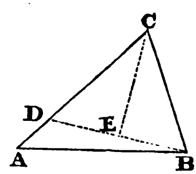
Suppose another line AE also equal to AC. Since AC is equal AB, by P. 5, ang. ACB is equal to ang. ABC; but in triangle AEC, the ext. ang. ACB, by P. 16, is greater than the int. ang. AEC, and ABC, being equal to ACB, is greater than AEC; therefore, by P. 19, the side AE is greater than BA, or than CA: thus it is proved that another line besides AB and AC cannot be drawn from A to BE equal to those lines.

3. *All heavy bodies free to move, continually descend, or seek the point which is nearest to the earth's centre.*



Let HALM represent the earth's circumference, HCL the rational horizon of station A, and AC a perpendicular to the centre; also let AB be a plane parallel to HL,—suppose it the channel of a canal; then water poured in at B will flow towards A, because at A by P. 19, the distance AC is less than the distance BC. A canal thus constructed might be full at A, and almost empty at B. In the same way a sphere placed at B would roll towards A, and finally settle or come to rest at A, because AC is the shortest line to the earth's centre.

4. *By aid of the* 19*th and the* 4*th Propositions we can construct a triangle when the base* AB, *the less angle,* A, *at the base, and the difference,* AD, *of the sides, are given.*



On the line AD, produced indefinitely, and forming with AB the ang. A, take AD, the difference of the sides; join D and B, and bisect DB in E; at E raise a perpendicular till it meets AC in C, and join CB; the triangle ABC is the triangle required.

By Hyp. and P. 19, BC is the least side, being opposite to the least ang. A. By Const., DE is equal to EB, EC common, and the angles at E equal; therefore, by P. 4, the side DC is equal to the side BC; and CA is greater than CB by the given difference AD. The figure ABC therefore is the triangle required.

Prop. 20.—Theor.

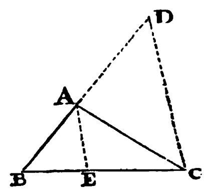
Any two sides of a triangle are together greater than the third side.

Construction.—Pst. 2. A st. line may be lengthened out in a st. line.
P. 3. From the greater line to cut off a part equal to the less.
Pst. 1. A st. line may be drawn from one point to another.

Demonstration.—P. 5. The angles at the base of an isosceles triangle are equal.

Ax. 9. The whole is greater than its part.

P. 19. The greater angle of every triangle is subtended by the greater side.



Exp.	1	Hyp.	Let ABC be a triangle;
	2	Concl. 1.	then the sides AB and AC are > BC;
		" 2.	sides AB and BC > AC;
		" 3.	" BC and CA > AB.
Cons.	1	Pst. 2, P. 3.	Produce BA and make AD = AC;
	2	Pst. 1.	join D and C.
Dem.	1	C. 1 & P. 5.	∵ AD = AC, ∠ADC = ∠ACD;
	2	C. 1 & 2.	but ∠BCD is > ∠ACD;
	3	Ax. 9.	∴ ∠BCD is > ∠ADC or BDC;
	4	P. 19.	and ∴ also BD is > BC.
	5	C. 1.	But BD = BA and AC together;
	6	D. 4 & 5.	∴ BA and AC together are > BC.
	7	Sim.	In like manner it may be proved that AB and BC are > AC,
	8	"	and BC and AC > AB.
	9	Conc. & R.	*Therefore any two sides of a triangle, &c.* Q.E.D.

Or,

Cons.		P. 9.	Let AE bisect ∠BAC.
Dem.	1	P. 16.	∵ ∠BEA is > ∠EAC, and ∠CEA than ∠EAB;
	2	P. 19.	∴ BA is > BE, and AC than CE:
	3	Concl.	And BA and AC > BE and CE, which are equal to BC.
	4	Recap.	*Therefore any two sides, &c.* Q.E.D.

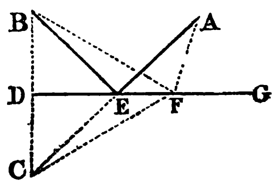
Cor.—*The difference of any two sides of a triangle is less than the remaining side.*

Dem.	1	P. 20.	The sides AC and BC are > AB;
	2	Sub.	take away AC both from (AC + BC), and also from AB;
	3	Ax. 5.	∴ BC is > AB diminished by AC, or than the difference between AB and AC.

N.B. In the proof of this Corollary, says *Lardner*, p. 32, "we assume something more than is expressed in the fifth Axiom. For we take for granted, that if one quantity (a) be greater than another quantity (b), and that equals be taken from both, the remainder of the former (a) will be greater than the remainder of the latter (b). This is a principle which is frequently used, though not expressed in the Axiom."

SCHOLIUM.—"Let the beginner remember that the object of this proposition is not to *convince* him of the truth stated, but to show how it may be connected with and deduced from the fundamental axioms and definitions."—*Mason*, p. 56

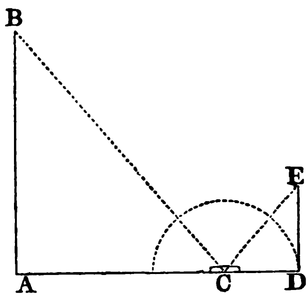
USE AND APPLICATION.—1. *Of all lines that can be drawn from one point*, A, *to another*, E, *and reflected to a third point*, B, *those are the shortest*, AE, EB, *which make the angle of incidence*, AEG, *equal to the angle of reflection*, BED.



CONST.	1	P. 12 & Pst. 2.	From B draw BD perp. to DG, and produce BD indefinitely;
	2	P. 3.	make DC = DB, and join EC.
	3	Assum., Pst. 1.	Also in DG assume another point F, and join AF, BF, and CF.
DEM.	1	C. 1, 2.	In triangles BED, CED, DE is common, BD equal to DC, and the angles at D equal.
	2	P. 4.	∴ BE is = CE, and ∠ BED = ∠ CED.
	3	Sim.	In the same way we prove BF to be = CF.
	4	Hyp. & C. & D. 2	Now ∠s BED, DEC, and AEG are all equal;
	5	C.	and AEC is a st. line.
	6	D. 5 & C. 3, P. 20	Also AFC is a △; ∴ AF and FC > AC:
	7	D. 2, 3.	but AC = AE and EB and FB = FC;
	8	Concl.	∴ AF, FC, or AF, FB, > AE and EB.
	9	Recap.	*Wherefore, of all lines that can be drawn*, &c. Q.E.D.

2. We may observe that natural causes act by the shortest lines; therefore all reflections are made by the lines which cause the angle of reflection to equal the angle of incidence. Hence, *by means of a mirror placed horizontally, we may construct a triangle the perpendicular of which shall be representative of the height of any object.*

Let AB be the height of a tower, AC a horizontal line, with a mirror at C. BCA will be the ang. of incidence of the light from B on the mirror at C: if now an observer at D measures the angle of reflection ECD, and the distance AC, he will have the means of constructing a triangle representative of ACB: for make the ang. ECD the angle of reflection, let CD contain the equal parts from a scale that represent the yards or feet in AC, and at D let a perpendicular DE be raised,—DE applied to the same scale of equal parts, will give the height of AB.



3. Were we to take as a Lemma, Prop. 4, bk. vi., that "the sides about the equal angles of equiangular triangles are proportionals," we should have from the foregoing construction the proportion, and consequently the equation, which would determine the height AB. For triangles BAC and EDC being similar, the base of the one is to the base of the other as the perpendicular of the one to the perpendicular of the other: or,

DC : AC :: ED : BA, and the product of the extremes equalling the product of the means,

$DC . BA = AC . ED$; whence $AB = \frac{AC . ED,}{DC,}$ or, $\frac{50 \times 6}{10} = 30$; when $AC = 50$, $ED = 6$, and $DC = 10$.

Prop. 21.—Theor.

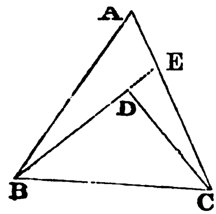
If from the ends of the side of a triangle there be drawn two st. lines to a point within the triangle, these lines shall be less than the other two sides of the triangle, but shall contain a greater angle.

Construction.—Pst. 2. A terminated st. line may be produced.

Demonstration.—P. 20. Any two sides of a triangle are together greater than the third side.

Ax. 4. If equals be added to unequals, the sums will be unequal.

P. 16. If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.



Exp.	1	Hyp.	Given the $\triangle$ ABC, and from B & C the st. lines BD, CD meeting within the $\triangle$ in the pt. D;
	2	Concl. 1.	then BD and CD shall be < BA and CA;
		" 2.	but the $\angle$ BDC shall be > $\angle$ BAC.
Cons.	1	Pst. 2.	Produce BD to meet CA in the point E.
Dem.	1	P. 20.	In $\triangle$ ABE the sides AB and AE are > BE;
	2	Add.	to each of these add the line EC;
	3	Ax. 4.	then the lines AB, AE, and EC, are > BE and EC.
	4	P. 20.	Again, in $\triangle$ CED, the sides CE and ED are > CD;
	5	Add.	to each of these add the line DB;
	6	Ax. 4.	then CE, ED, and DB together are > CD and DB together.
	7	D. 3.	But AB and AC are > BE and EC;
	8	*à fort.*	much more are AB and AC > CD and DB.
	9	P. 16.	Again, in $\triangle$ CDE the ext. $\angle$ BDC is > the int. $\angle$ CED,
	10	P. 16.	and in $\triangle$ ABE the ext. $\angle$ CED is > the int. $\angle$ BAC;
	11	*à fort.*	much more is $\angle$ BDC > $\angle$ BAC.
	12	Recap.	*Therefore, if from the ends of a side,* &c.

Q.E.D.

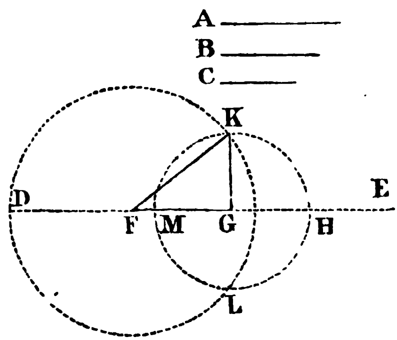
Use and Application.—In Optics this proposition is used to prove that if from A we could see the line BC, and also from D a point nearer to the line, the base BC would appear less from A than from D: it does this on the principle that quantities seen under a greater angle appear greater. For this reason the apparent diameter of the sun measures more when the earth is in *perihelion*, than when it is in *aphelion*. And thus,—according to Vitruvius, who composed his work on Architecture, about 15 B.C.—the tops of very high pillars should be made but little tapering, because they will, from the distance, of themselves seem less.

PROP. 22.—PROB.

To make a triangle, of which the sides shall be equal to three given st. lines, but any two whatever of these must be greater than the third.

SOLUTION.—P. 3. From the greater line to cut off a part equal to the less.
Pst. 3. A circle may be drawn with any radius.
Pst. 1. A line may be drawn from point to point.

DEMONSTRATION.—Def. 15. A circle is a figure bounded by one continued line, called its circumference, and having a certain point within it from which all lines drawn to the circumference are equal.
Ax. 1. Magnitudes which are equal to the same magnitude, are equal to each other.



EXP.	1	Data.	Given the st. lines A, B, and C, of which A and B > C, A and C > B, and B and C > A;
	2	Quæs.	to make a △ with sides = A, B, and C.
CONS.	1	Assum.	Assume a st. line DE unlimited towards E;
	2	P. 3.	make DF = A, FG = B, and GH = C;
	3	Pst. 3.	from F with DF draw the ⊙ DKL,
	4	„ 3.	and from G with GH the ⊙ HLK;
	5	„ 1.	join the points FK and GK;
	6	Sol.	then the fig. KFG is a △, having FK = A, FG = B, and GK = C.
DEM.	1	C.3, Def. 15	∵ F is the centre of ⊙ DKL, FK = FD;
	2	C.2 & Ax. 1	but FD = A, ∴ FK also = A.

DEM.	3	C.3, Def.15	Again, ∵ G is the centre of ⊙ HKL, GH = GK;
	4	C.2 & Ax.1	but GH = C; ∴ also GK = C,
	5	C. 2.	and FG is equal to B;
	6	D. 2, 4, & 5.	∴ FK, FG, GK are respectively equal to A, B, and C.
	7	Recap.	*And therefore the* △ FKG *has its three sides, &c.* Q.E.F.

SCHOLIUM.—1. In this proposition it is assumed that *the two circles will have at least one point of intersection.*

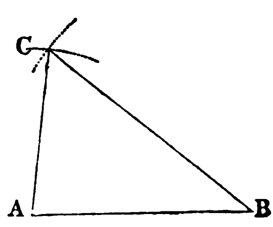
DEM.	1	C. & Dat.	∵ FG and GK are > FK, FH > the rad. of DLK.
	2		∴ the point H is outside of the ⊙ DLK.
	3	C. & Dat. Sub	Also, ∵ FK and GK are > FG, take away GK or GH;
	4		∴ FM is < FK, and M is within the ⊙ DLK.
	5	D. 2 & 4.	Now, H being without, and M within, the ⊙s,
	6	Concl.	therefore the circles intersect.

2. If two of the given lines were together equal to the third, the circles would touch externally; if the two were together less than the third, the circles would not touch at all: in either case no triangle could be drawn.

USE AND APPLICATION.—1. All rectilineal figures being divisible into triangles, this Proposition is of very extensive use in the construction of Geometrical figures,—either for making one rectilineal figure *equal* to another, or on the theory of Representative Values making one figure *like to* another: in the *first* case the triangles into which the rectilineal figure has been divided are repeated, side for side, in another rectilineal figure of exactly the same linear dimensions: and when the construction is completed, if the one figure were placed on the other, the two would correspond, angle to angle, line to line, and point to point: in the *second* case, that of making one rectilineal figure similar to another; the sides and angles of the first must be measured, and from a scale of equal parts, lines drawn in the second, *representative* of those in the first, and angles in the second equal to those in the first;—for *equality of angles*, according to Definition 1, book vi., is essential to *similarity of figure.*

2. Practically, a triangle with sides equal to three given lines, will be drawn, by describing arcs, with radii equal to the sides, intersecting in C, and joining CA, and CB; thus,

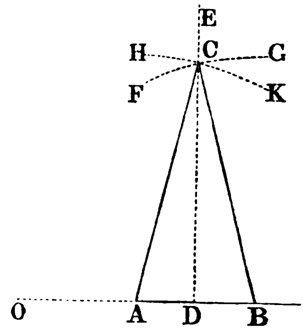
Given, three lines AB *equal to* 20, BC *to* 25, *and* CA *to* 15; *to form a triangle.*



From the scale set off a line AB equal to 20; at A with the distance CA, 15, desc. an arc; and at B with the distance BC, desc. another arc: the arcs both intersect in C, and joining the points, ABC is the triangle required.

3. ***On a given line*, AB, *to describe an isosceles triangle*, AEB, *having each of the equal sides*, AC, BC, *double of the base*, AB, *or equal to* BO.**

CONS.	1	P.10, 11, Pst. 2	Bisect AB in D, by the perp. DE, produced indefinitely to E.
	2	Pst. 3, P. 3.	From A with rad. equal to twice AB, desc. the arc FCG;
	3	„ 3, „ 3.	and from B with the same rad. desc. the arc HCK;
	4	„ 1.	join the points CA and CB.
	5	Sol.	The fig. ACB is the isosc. △ required.
DEM.	1	C. 1.	∵ AD = DB, DC common, and ∠ADC = ∠BDC.
	2	P. 4.	∴ the line AC equals the line BC,
	3	C. 2 & 3.	and they are each double of AB.
	4	Conc.	Wherefore on a given line AB, &c, Q.E.F.



4. *On a given line*, AB, *and with a given side equal to* BO, *to describe an isosceles triangle.*

The same construction being made, and the same demonstration followed, we arrive at the conclusion,—that BC equals CA, and that each of them is equal to the given side BO.

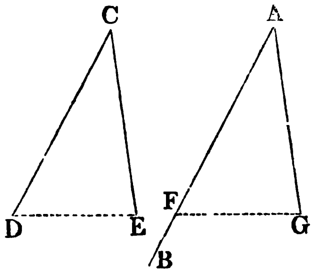
—

PROP. 23.—PROB.

At a given point in a given line to make a rectilineal angle equal to a given rectilineal angle.

SOLUTION.—Pst. 1. A st. line may be drawn from one point to another.
P. 22. With three given lines, any two of which are greater than the third, to make a triangle.

DEMONSTRATION.—P. 8. If two triangles have the three sides of one equal to the three sides of another, each to each, the angle contained by two equal and conterminous sides of the one, shall be equal to the angle contained by any two equal and conterminous sides of the other.

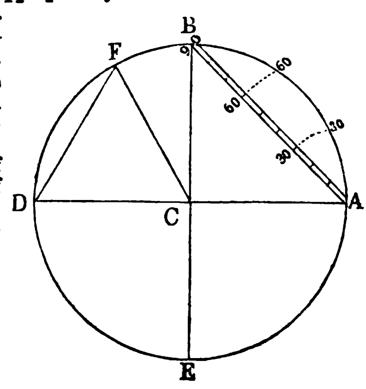


Exp.	1	Data.	Given the point A in the line AB, and ∠ DCE;
	2	Quæs.	required at A in AB to make an ∠ = DCE.
Cons.	1	Assum. & Pst. 1.	In CD, DE take points D and E, and join DE;
	2	P. 22.	with AF = CD, AG = CE, and FG = DE, make the △ AFG;
	3	Sol.	then the ∠ FAG shall be = ∠ DCE.
Dem.	1	C. 2.	∵ FA = DC, AG = CE, and DE = FG;
	2	P. 8.	∴ the ∠ FAG = ∠ DCE.
	3	Recap.	*Therefore, at point* A *in* AB, &c. Q.E.F.

Scholium.—This Proposition is an extension, or rather a generalization, of the eleventh: by the eleventh we draw an angle of a particular species,—a right angle; but by the twenty-third we draw any angle whatever.

Use and Application.—1. This Proposition is of the widest use, in Surveying, Engineering, Perspective, and indeed in all the other parts of Practical Mathematics.

2. Next to the use of the Semicircle for measuring and making angles of a determinate magnitude, that of a *Line of Chords* is most important. Its construction and use we may here appropriately introduce.



Take a circle and draw a diameter DA; bisect DA in C, and at C draw CB at rt. angles to DA, and produce BC to E;—the circle will be divided into four quadrants, each 90°. Let the arc AB of 90° be divided into arcs of 10°, 20°, 30°, &c., up to 90. Draw AB, the chord of 90°, and from A, as a centre, set off on AB the chords of 10°, 20°, 30°, &c.: the divisions on the line AB will be a line of chords.

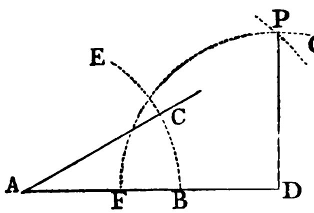
N.B. *The chord of* 60° *is equal to the radius of the circle.*

In triangle DCF, let the angle DCF measure 60°,—then, since CF and CD are equal, the angles CDF

and CFD will be equal to each other, each measuring 60°. The triangle, therefore, being equiangular, is also equilateral; and DF, the chord of 60°, equals the radius DC.

On this property depends the use of the Line of Chords in the following Problems.

PROB. I.—***To make an angle to contain a certain number of degrees, as* 30°.**



Draw AD an indefinite line; from A with the radius equal to the chord of 60° describe an arc BE; and from the line of chords take 30°, and set the distance from B on the arc BE;—BC will be the arc of 30°, and CAB the angle required.

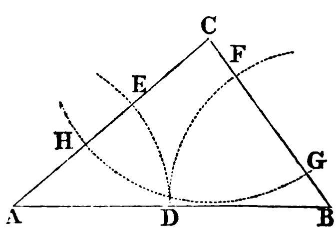
PROB. II.—***An angle being given,*** CAB, ***to find the measure of it in degrees of a circle.***

With the chord of 60° for radius, describe the arc BC; take the distance BC in the compasses, and apply the distance to the line of chords;—the number of degrees in the arc BC will be ascertained.

PROB. III.—*From the extremity,* D, *of a line* AD, *to draw a perpendicular.*

From D, with the chord of 60° for radius, describe the arc FG; take the chord of 90° for a radius, and from F describe another arc intersecting FG in P; draw PD,—and it is the perpendicular required.

PROB. IV.—*To construct a triangle of which the base* AB *contains* 30 *equal parts, the angle at* A 40°, *and the side* AC 25 *equal parts; and to find the angles* C *and* B, *and the other side* BC.



Take 30 from the scale of equal parts, and by P. 3, draw AB; at A, with the chord of 60°, desc. the arc DE, and from D, with the chord of 40°, or of ang. A, cut the arc in E; join AE,—and EAB is the required angle. Produce AE, and set on it 25, or AC; join CB,—and the triangle is completed.

To measure CB, take the distance from C to B, and apply it to the scale, which will give the equal parts 19: and to measure the angles,—with the chord of 60°, describe from C and B the arcs GH and DF, take in the compasses the distances G to H, and D to F,—these distances applied to the line of chords will give the angle C 85°, and the angle B 55°.

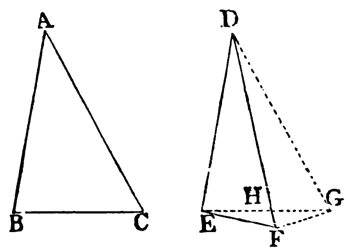
—

PROP. 24.—THEOR.

If two triangles have two sides of the one equal to the two sides of the other, each to each, but the angle contained by the two sides of the one greater than the angle contained by the two sides of the other, the base of that which has the greater angle shall be greater than the base of the other.

CONSTRUCTION.—P. 23. To make an angle equal to a given angle.
P. 3. From the greater line to take a part equal to the less.
Pst. 1. Any two points may be joined by a st. line.

DEMONSTRATION.—P. 4. When two sides and the included angle of one triangle are equal to the two sides and the included angle of another triangle, the triangles are equal in every respect.
P. 5. The angles at the base of an isosceles triangle are equal.
Ax. 9. The whole is greater than its part.
P. 19. The greater angle of every triangle is subtended by the greater side.



EXP.	1	Hyp. 1.	In $\triangle$s ABC, DEF, let AB = DE, and AC = DF;
	2	„ 2.	but $\angle$ BAC > $\angle$ EDF;
	3	Concl.	then the side BC > the side EF.
CONS.	1	Hyp. & P. 23	Let DF be $\not>$ DE, and make $\angle$ EDG = $\angle$ BAC;
	2	P. 3, & Pst. 1	also make DG = AC or DF, and join EG, GF.
DEM.	1	Hyp. & C. 1.	$\because$ AB = DE, AC = DG, & $\angle$ BAC = $\angle$ EDG,
	2	P. 4.	$\therefore$ EG is equal to BC.
	3	C. 2 & P. 5.	And $\because$ DG = DF, $\therefore$ $\angle$ DGF = $\angle$ DFG.
	4	Ax. 9.	But $\angle$ DGF is > EGF, and $\angle$ DFG > $\angle$ EGF;
	5	*à fort.*	much more is $\angle$ EFG > $\angle$ EGF.

DEM.	6	P. 19.	But the greater angle is opposite to the greater side;
	7	D. 5 & P. 19	∴ the side EG is > EF.
	8	D. 2.	But EG = BC, and ∴ BC is > EF.
	9	Recap.	*Therefore, if two triangles have two sides, &c.* Q.E.D.

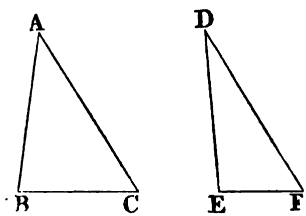
SCHOLIUM.—In this proposition it is assumed that D and F will be on different sides of EG; or in other words that DH is less than DF or DG.

—

PROP. 25.—THEOR.

If two triangles have two sides of the one equal to the two sides of the other, each to each, but the base of one greater than the base of the other, the angle contained by the sides of the one which has the greater base shall be greater than the angle contained by the sides equal to them of the other.

DEMONSTRATION.—P. 4. If two triangles have two sides and the included angle in each equal, the triangles are equal in every respect.
P. 24. If two triangles have two sides equal in each, but of the included angles one greater than the other, the base of that which has the greater angle shall be greater than the base of the other.



EXP.	1	Hyp. 1.	In △s ABC, DEF, let AB = DE, & AC = DF,
	2	„ 2.	but the side BC > EF;
	3	Concl.	then the ∠ BAC is > ∠ EDF.
DEM.	1	Sup.	If ∠ BAC is ≯ ∠ EDF, it is either equal or less:
	2	Sup.	Suppose that ∠ BAC = ∠ EDF;
	3	Hyp. & P. 4	then BC = EF:

DEM.	4	Hyp.	But BC is $\neq$ EF;
	5	Conc.	$\therefore$ $\angle$ BAC is $\neq$ $\angle$ EDF.
	6	Sup.	Again, suppose $\angle$ BAC $<$ $\angle$ EDF;
	7	H. & P. 24.	then BC is $<$ EF:
	8	Hyp.	But BC is $\nless$ EF;
	9	Concl.	$\therefore$ $\angle$ BAC is $\nless$ $\angle$ EDF.
	10	D. 5 & 9.	Now, $\angle$ BAC is $\neq$, nor $<$, EDF;
	11	Concl.	$\therefore$ $\angle$ BAC is $>$ $\angle$ EDF.
	12	Recap.	*Therefore, if two triangles have two sides,* &c. Q.E.D.

SCHOLIUM.—Propositions 24 and 25 have the same relation to each other as Props. 4 and 8, and the four may be combined thus:—*If two triangles have two sides of the one respectively equal to two sides of the other, the remaining side of the one will be greater or less than, or equal to, the remaining side of the other, according as the angle opposed to it in the one is greater or less than, or equal to, the angle opposed to it in the other; or vice versâ.*—LARDNER'S *Euclid*, p. 56.

USE AND APPLICATION.—The principal use of Props. 24 and 25 is to assist in the demonstration of the following propositions.

PROP. 26.—THEOR.—*(Important.)*

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz., either the sides adjacent to the equal angles in each, or the sides opposite to them, then shall the other sides be equal, each to each, and also the third angle of the one to the third angle of the other.

CONSTRUCTION.—P. 3. From the greater of two lines to cut off a part equal to the less.

Pst. 1. A line may be drawn from one point to another.

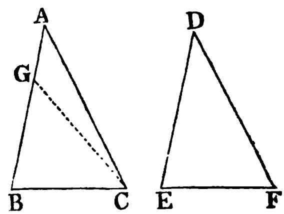
DEMONSTRATION.—P. 4. Two triangles are equal, when two sides and the included angle in one are equal to two sides and the included angle in the other.

Ax. 1. Magnitudes which are equal to the same, are equal to each other.

P. 16. If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

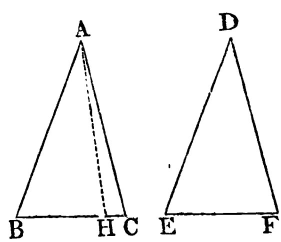
EXP.	1	Hyp. 1.	In $\triangle$s ABC, DEF, let $\angle$ ABC = $\angle$ DEF, and $\angle$ ACB = $\angle$ DFE;
	2	" 2.	also let one side = one side;
	3	Concl. 1.	then the other sides shall = the other sides.
	4	" 2.	and the third $\angle$ of the one = the third $\angle$ of the other.

CASE I.—*Let the equal sides be adjacent to the equal angles.*



EXP.	1	Hyp.	Let side BC = side EF;
	2	Concl. 1.	then AB = DE, and AC = DF,
	3	" 2.	and ∠ BAC = ∠ EDF.
CONS.	1	Sup.	For if AB is ≠ DE, one is > the other.
	2	Sup.	Let AB be > DE:
	3	P.3 & Pst.1	Make BG = DE, and join GC.
DEM.	1	C.3 & Hyp.	∵ in △s GBC, DEF, BG = ED, BC = EF, and ∠ GBC = ∠ DEF;
	2	P. 4.	∴ GC = DF, △ GBC = △ DEF, and ∠ BCG = ∠ DFE:
	3	Hyp.	But ∠ DFE = ∠ BCA;
	4	Ax. 1.	∴ ∠ BCG = ∠ BCA, the less = the greater, which is impossible.
	5	Concl.	∴ AB is not ≠ DE, *i. e.*, AB does = DE;
	6	D.5 & Hyp.	Hence in △s ABC, DEF, AB = DE, BC = EF, and ∠ ABC = ∠ DEF;
	7	P. 4.	∴ AC = DF, and ∠ BAC = ∠ EDF.

CASE II.—*Let the equal sides be opposite the equal angles in each.*



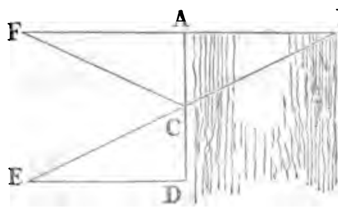
EXP.	1	Hyp.	Let AB = DE, ∠ B = ∠ E, and ∠ ACB = ∠ DFE;
	2	Concl. 1.	then AC shall = DF, BC = EF,
	3	" 2.	and ∠ BAC = ∠ EDF.
CONS.	1	Sup.	For if BC ≠ EF one of them is the greater.
	2	Sup.	Let BC be greater than EF;
	3	P.3, Pst. 1	Make BH = EF, and join AH.
DEM.	1	C. & Hyp.	∵ in △s ABH, DEF, BH = EF, AB = DE, and ∠ ABH = ∠ DEF;

DEM.	2	P. 4.	∴ $AH = DF$, $\triangle ABH = \triangle DEF$, and $\angle BHA = \angle EFD$:
	3	Hyp.	But $\angle EFD = \angle BCA$;
	4	Ax. 1.	∴ $\angle BCA = \angle BHA$, or the int. = the ext. angle;
	5	P. 16.	which is impossible:
	6	Concl.	∴ BC is not $\neq$ EF, *i. e.*, $BC = EF$.
	7	D.6 & Hyp	Hence in $\triangle$s ABC, DEF, $AB = DE$, $BC = EF$, and $\angle ABC = \angle DEF$;
	8	P. 4.	∴ $AC = DF$, and $\angle BAC = \angle EDF$.
	9	Recap.	*Wherefore, if two triangles have two angles of the one*, &c. Q.E.D.

SCHOLIUM.—1. This is the last of the *three criteria* for inferring the equality of triangles; the *first* criterion, contained in Prop. 4, is—*that the two sides and the included angle of each triangle, be equal;* the *second*, in Prop. 8, *that the three sides of one triangle equal the three sides of the other;* and the *third*, in Prop. 26, *that in each triangle two angles be equal and one side, either adjacent to the equal angles, or opposite to one of them.*

2. Of the six parts of a triangle any three, except the three angles, being given equal to one another, the equality of the other parts may be readily proved.

USE AND APPLICATION.—1. The discovery of this Proposition is ascribed to THALES, who made use of it *for measuring inaccessible distances*, as from A to B.



At the station A set out a line AC, perpendicular to AB; and at C observe the angle ACB, and by P. 23 make the ang. ACF equal to ang. ACB; the lines BA and CF produced meet in the point F; and AF will be found equal to AB.

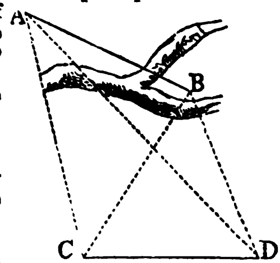
For in triangles ACB, ACF, by construction, we have the angles at A equal,—the angles ACB and ACF equal,—and the side AC common; therefore by P. 26, AF will equal AB; if now we measure AF, we have a measurement equal to that of AB.

Another construction might be used, by producing AC to D, and making CD equal to CA; then at D draw DE perp. to AD, and produce BC, till it cuts DE in E; in this case DE may be proved equal to the distance from A to B.

2. *By the Theory of Representative Values the distance of two stations* A *and* B *may be found.* Suppose AC to measure 70 feet or yards, ang. A 90°, ang. ACB 66°; at A make a right angle, and on AC set off 70 equal parts; at C draw an angle of 66°; the line CB will cut off B, the distance from A: let the distance from A to B be applied to the same scale, and AB will be found equal to about 150 feet or yards, according to the unit of length fixed on for AC.

3. ***When neither* A *nor* B *are accessible*,** a process rather different has to be employed, founded however on strictly geometrical principles.

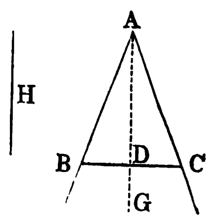
Select two stations, C and D, from both of which A and B are visible; and measure the line CD, suppose 150; take the angles BCD 60°, and BDA, $22\frac{1}{2}$°; ADC, 45°; and ACB, 45°: it is required to ascertain the distance AB.



From a scale of equal parts make CD 150; draw the angles BCD 60°, and BDA $22\frac{1}{2}$°; the lines CB and DB meet in B: again draw the angles ACB 45°, and ADC 45°; the lines DA and CA meet in A: take the distance AB in the compasses, and apply it to the same scale;—AB will be found to be about 158.

4. *Given the vertical angle* A, *and the perpendicular height* H, *of an isosceles triangle, to construct it.*

Cons.	1	P. 9 & 3.	Bisect ∠ A by AG, and cut off AD = H;
	2	P. 23.	through D draw BC at ⊥ to DA:
	3	Sol.	the figure ABC is the isosc. △ required.
Dem.	1	C. 1 & 2.	∵ ∠BAD = ∠CAD, ∠BDA = ∠CDA,
	2	C.	and AD, which is equal to H, common;
	3	P. 26, Def. 25	∴ AB = AC, and ABC is an isosc. △.
	4	Recap.	*Therefore, with the vertical angle* A *given*, &c.

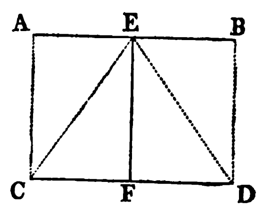


Q. E. F.

Lemma.

A Line which is Perpendicular to one Parallel, is also Perpendicular to the other.

Exp.	1	Hyp.	Let AB be ∥ CD, and EF perpen. to CD;
	2	Concl.	then EF shall also be perpen. to AB.
Cons.	1	P. 3 or 10.	Make CF = FD;
	2	P. 11.	at C and D erect perpendiculars CA and DB,



CONS.	3	D. 35, Ax. 12	which by the note to the definition of parallels equal E F;
	4	Pst. 1.	and draw E C, E D.
DEM.	1	C. & Hyp.	In △ C E F, D E F, the line C E = F D, F E common, and ∠ C F E = ∠ D F E;
	2	P. 4.	∴ E C = E D, ∠ F E D = ∠ F E C, and ∠ F D E = ∠ F C E;
	3	Sub.	take away the equals E C F and E D F from the equals A C F and B D F;
	4	Ax. 3.	∴ the rem. ∠ E C A = the rem. ∠ E D B.
	5	P. 4.	Now △s C A E, D B E shall have ∠ C E A = ∠ D E B;
	6	Add.	Adding the equal angles D E B, C E A to the equals D E F, C E F,
	7	Ax. 2.	∴ the whole ∠ B E F = the whole ∠ A E F;
	8	Const.	and these angles are adjacent:
	9	Def. 10.	∴ E F is also perpendicular to A B.
	10	Recap.	*Wherefore a line which is perpendicular*, &c.

Q.E.D.

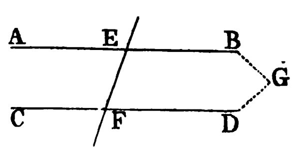
PROP. 27.—THEOR.

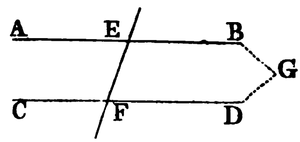
If a st. line falling upon two other st. lines makes the alternate angles equal to one another, these two lines shall be parallel.

CONSTRUCTION.—Pst. 2. A st. line may be produced indefinitely in a st. line.

DEMONSTRATION.—P. 16. If one side of a triangle be produced, the ext. angle is greater than either of the interior opposite angles.
Def. 35. Parallel st. lines are such as are in the same plane, and which being produced ever so far both ways do not meet.

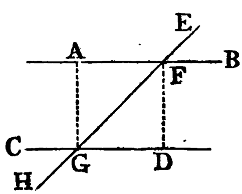
EXP.	1	Hyp. 1.	Let E F fall on the two st. lines A B and C D,
	2	„ 2.	and make the alt. ∠ A E F = the alt. ∠ E F D;
	3	Concl.	then the st. line A B is ∥ C D.





CONS.	1	Sup.	If AB and CD are not ‖,
	2	Def. 35.	on being produced they will meet towards A, C, or B, D;
	3	Pst. 2 & Sup.	let them be produced and if possible meet in G, and form the △ EFG.
DEM.	1	C. 3 & P. 16	∵ GEF is a △, the ext. ∠ AEF > the int. ∠ EFG;
	2	Hyp.	but ∠ AEF = ∠ EFG;
	3	D. 1 & 2.	∴ ∠ AEF is both > and = ∠ EFG;
	4	*ad imposs.*	which is impossible:
	5	Concl.	∴ AB, CD do not meet towards B, D.
	6	Sim.	In the same way it may be shown, they do not meet towards AC:
	7	Def. 35.	∴ AB is parallel to CD.
	8	Recap.	*Wherefore, if a st. line falling, &c.* Q.E.D.

SCHOLIUM.—Since, however, there are some curved lines which are not parallels, though they never intersect, as in the Introduction, § vi., p. 28, another demonstration may be made.



EXP.	1	Hyp.	Let EH fall on AB, CD, and make the alt. ∠s AFG and FGD equal;
	2	Concl.	then the lines AB and CD are parallel.
CONS.	1	P. 12.	At G draw GA perpend. to AB;
	2	P. 3, Pst. 1	take GD = AF, & join FD.
DEM.	1	C. 2.	In △s AGF, DFG, GD = AF, GF common,
	2	Hyp.	and ∠ AFG = ∠ FGD;
	3	P. 4.	∴ AG = DF, ∠ GDF = ∠ GAE:
	4	C. 1.	But ∠ GAF is a rt. angle,
	5	Def. 10.	∴ GDF is a rt. ang., and DF perp. to CD.
	6	Hyp.	Moreover the par. to CD is to be drawn from F;
	7	D. 3.	and since the perpendiculars AG and DF are equal,
	8	Ax. 12, *n.*	the parallel to CD must pass through A:
	9	Concl.	∴ AB is parallel to CD. Q.E.D.

SCHOLIUM.—Of the angles which two st. lines, one at each extremity of a third line, make with it, the alternate angles are those which are on opposite sides and at opposite extremities of the third line.

PROP. 28.—THEOR.

If a st. line falling upon two other st. lines makes the exterior angle equal to the interior and opposite angle upon the same side of the line; or makes the interior angles upon the same side together equal to two rt. angles; the two st. lines shall be parallel.

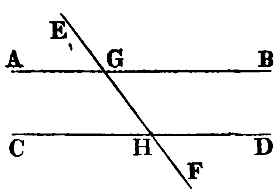
DEMONSTRATION.—P. 15. If two st. lines cut one another, the vertical angles shall be equal.

Ax. 1. Magnitudes equal to the same are equal to one another.

P. 27. If a st. line falling upon two other st. lines makes the alternate angles equal, these two st. lines shall be parallel.

P. 13. The angles which one st. line makes with another upon one side of it, are equal to two rt. angles.

Ax. 3. If equals be taken from equals, the remainders are equal.



EXP.	1	Hyp. 1.	Let the st. line EF fall on the two st. lines AB, CD;
		„ 2.	make the ext. ∠ EGB = the int. & opp. ∠ GHD;
		„ 3.	and the int. ∠s BGH + GHD = two rt. angles;
	2	Concl.	then the st. line AB is ∥ the st. line CD.
DEM.	1	Hyp.	For, ∵ ∠ EGB = ∠ GHD,
	2	P. 15.	and ∠ EGB = ∠ AGH;
	3	Ax. 1.	∴ ∠ AGH = ∠ GHD:
	4	P. 27.	and ∴ also st. line AB is ∥ CD.
	5	Hyp.	Again, ∵ ∠s BGH + GHD = two rt. angles,
	6	P. 13.	and ∠s AGH + BGH = two rt. angles;
	7	Ax. 1.	∴ ∠s AGH + BGH = ∠s BGH + GHD:
	8	Sub.	Take away the common ∠ BGH,
	9	Ax. 3.	∴ the rem. ∠ AGH = the rem. ∠ GHD:
	10	Cons.	But ∠s AGH and GHD are alternate angles;
	11	P. 27.	∴ the st. line AB is ∥ the st. line CD.
	12	Recap.	*Therefore, if a st. line falling upon two other,* &c. Q.E.D.

SCHOLIUM.—1. When one st. line cuts, or falls upon, two other st. lines, —*of the four angles* formed on each side of the incident line, *taken in succession*, each pair consists of an ext. and an int. adjacent angle; but *taken alternately*, each pair consists of an ext. and an int. opposite angle.

2. The principle really assumed in the twelfth Axiom is,—that two st. lines which intersect each other cannot both be parallel to the same st. line.

Prop. 29.—Theor.

If a line fall upon two parallel st. lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite angle upon the same side; and likewise the two interior angles upon the same side together equal to two rt. angles.

Demonstration.—Ax. 4. If equals be added to unequals, the wholes are unequal.

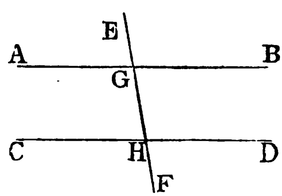
P. 13. The angles which one st. line makes with another upon one side of it are either two rt. angles, or are together equal to two rt. angles.

Ax. 12. If a st. line meets two st. lines, so as to make the two int. angles on the same side of it together less than two rt. angles, these st. lines, being continually produced, shall at length meet upon that side on which are the angles less than two rt. angles.

Def. 35. Parallel lines are such as are in the same plane, and which being produced ever so far both ways do not meet.

P. 15. If two st. lines cut one another, the vertical angles are equal.

Ax. 1. Magnitudes equal to the same, are equal to each other.



Exp.	1	Hyp.	Let the st. line EF fall on the ‖s AB, CD;
	2	Concl. 1.	then the ∠ AGH = the alt. ∠ GHD;
		" 2.	the ext. ∠ EGB = the int. opp. ∠ GHD;
		" 3.	and the int. ∠s BGH + GHD = two rt. angles.
Dem.	1	Sup. 1.	If ∠ AGH is ≠ ∠ GHD, one is > the other:
	2	" 2.	Let ∠ AGH be > ∠ GHD;
	3	Add.	and to each angle let the ∠ BGH be added;
	4	Ax. 4.	then ∠s AGH, BGH together are > ∠s BGH, GHD:
	5	P. 13.	But ∠s AGH, BGH together are = two rt. angles;
	6	Ax. 1.	∴ ∠s BGH, GHD together are < two rt. angles.
	7	Ax. 12.	Now on this condition AB and CD, being produced, will meet

DEM.	8	Hyp. Def. 35	but AB is parallel to CD;
	9	Concl. 1.	∴ ∠ AGH is not $\neq$, *i. e.*, is =, ∠ GHD:
	10	P. 15 & Ax. 1	But ∠ AGH = EGB, ∴ EGB = GHD;
	11	Add.	to each angle add ∠ BGH;
	12	Ax. 2.	then ∠ s EGB + BGH = ∠ s GHD + BGH:
	13	P. 13.	But ∠ s EGB + BGH = two rt. angles;
	14	Ax. 1.	∴ ∠ s GHD + BGH = two rt. angles.
	15	Recap.	*Wherefore, if a line fall on two parallel lines*, &c. Q.E.D.

SCHOLIUM.—1. This Proposition, depending on the 12th Axiom, is the converse of Props. 27 and 28.

2. It has been objected to the twelfth axiom that it is not self-evident, and that it forms the converse to P. 27: now, both the assumed axiom and its converse should be so obvious as not to require demonstration.

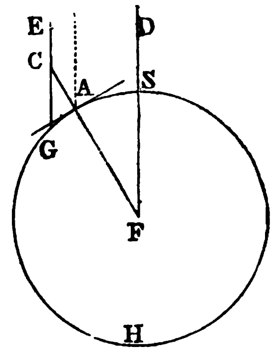
3. "The *Twelfth Axiom* may be expressed in any of the following ways: —Two diverging rt. lines cannot be both parallel to the same rt. line: or, If a rt. line intersect one of two parallel rt. lines, it must also intersect the other: or, Only one rt. line can be drawn through a given point parallel to a given rt. line."—LARDNER, p. 316.

4. The *least objectionable* definition of parallel st. lines is,—"Parallel lines are such as lie in the same plane, and which neither recede from nor approach to each other.—POTTS' *Euclid*, p. 50.

USE AND APPLICATION.—ERATOSTHENES, who died B.C. 196, at the age of 80, applied the last three Propositions to the measuring of the circumference of the earth, and followed the method still employed. *(See Dictionary of Greek and Roman Biography*, vol. ii., p. 44–47.) The principle with which he set out was, that two rays of light proceeding from the centre of the sun to two points on the earth, are physically parallel. He learned that "at Syene, in Upper Egypt, deep wells were enlightened to the bottom on the day of the summer solstice, and that vertical objects cast no shadows. He concluded, therefore, that Syene was on the tropic, and its latitude equal to the obliquity of the ecliptic, which he had determined to be 23° 51′ 20″: he presumed that it was in the same longitude as Alexandria, in which he was out about 3°, which is not enough to produce what would at that time have been a sensible error. By observations made at Alexandria, he determined the zenith of that place to be distant by the fiftieth part of the circumference from the solstice; which was equivalent to saying that the arc of the meridian between the two places is 7° 12′." The distance from Alexandria to Syene, Eratosthenes gives as 5000 stadia. From these data it is a very simple operation to deduce the earth's circumference; for if an arc of 7° 12′ measures 5000 stadia, the question to be solved, is,—How many stadia are there in 360°?

For the demonstration of the process, we suppose the circumference GSH to be representative of the earth's circumference; Alexandria to be at the point A, and Syene at the point S; at A the style AC is erected

perpendicular to the horizon: D S and E C are rays of light from the sun when in the Solstice, the ray D S perpendicular to Syene, and continued towards F, the earth's centre. Now at A, take the angle A C G, made by A C the perpendicular to the horizon, and by the ray E G; then, because the rays D F and E G are parallel, and the line C F meets them, the alternate angles A C G and S F A are equal. Thus, if we measure the one, we at the same time ascertain the other. The arc which measures angle S F A is S A; and if the actual distance in miles from S to A be measured, we have the elements for finding the earth's circumference; for as the degrees in the arc A S are to the distance measured from A to S, so are 360° to the measure in units of length of the whole circumference.



Prop. 30.—Theor.

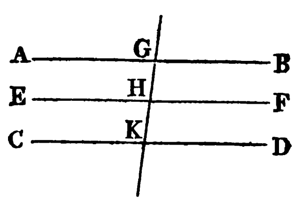
Straight lines which are parallel to the same st. line are parallel to each other.

Demonstration.—P. 29. If a st. line fall upon two parallel st. lines, it makes the alternate angles equal to one another; and the exterior angle equal to the int. and opp. angle upon the same side; and likewise the two int. angles upon the same side together equal to two rt. angles.

Ax. 1. Things equal to the same thing are equal to each other.

P. 27. If a st. line falling upon two other st. lines makes the alt. angles equal to one another, these two st. lines shall be parallel.

Exp.	1	Hyp.	Let AB and CD be each ∥ to EF;
	2	Concl.	then AB shall be ∥ to CD.
Cons.	1	Sup.	Let the line GHK cut the lines AB, EF, and CD.
Dem.	1	C. & Hyp.	∵ the line GHK cuts the ∥s AB, EF,
	2	P. 29.	∴ the $\angle$ AGH = $\angle$ GHF;
	3	C. & Hyp.	And, ∵ GK also cuts the ∥s EF, CD,
	4	P. 29.	∴ the $\angle$ GHF = $\angle$ GKD:



DEM.	5	D. 2.	But ∠ AGH or AGK = ∠ GHF;
	6	Ax. 1.	∴ ∠ AGK = ∠ GKD;
	7	P. 27.	and ∴ AB is ∥ to CD.
	8	Recap.	*Wherefore, straight lines which are parallel,* &c. Q.E.D.

SCHOLIUM.—1. The corollary to this proposition, that two lines parallel to the same line cannot pass through the same point, is equivalent to the twelfth axiom.

2. The first axiom and this proposition are similar.

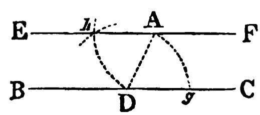
—

PROP. 31.—PROB.

To draw a st. line through a given point, parallel to a given st. line.

SOLUTION.—Pst. 1. A st. line may be drawn from one point to another.
P. 23. At a given point in a line to make an angle equal to a given angle.
Pst. 2. A terminated st. line may be produced to any length in a st. line.

DEMONSTRATION.—P. 27. If a st. line falling upon two other st. lines make the alternate angles equal, these two lines are parallel.

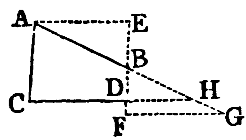


EXP.	1	Data.	Let A be the given point, & BC the given line;
	2	Quæs.	to draw through A a line ∥ to BC.
CONS.	1	Sup. & Pst. 1	In BC take any point D, and join A, D;
	2	P. 23.	At A in AD make the ∠ DAE = ∠ ADC;
	3	Pst. 2.	produce EA to F;
	4	Sol.	then EF, through A, is ∥ to BC.
DEM.	1	C. 1, 2.	∵ AD meets EF and BC, and makes ∠ EAD = alt. ∠ ADC,
	2	P. 27.	∴ EF is ∥ BC.
	3	Recap.	*Wherefore, through a given point,* &c. Q.E.F.

SCHOLIUM.—1. Practically, a parallel to BC through the point A, may be drawn with a parallel ruler, or with the triangular square. If the compasses are used, the way is *(see the last figure)*,—Join A and any point, as D, in BC: from D with DA describe the arc Ag, and from A with AD the

arc Dh; then, with the distance from g to A in the compasses, from D cut the arc Dh in h; join Ah and produce it;—EF is the parallel required.

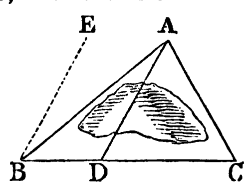
2. The *Twelfth Axiom*, that if a line falling on two other lines make the interior angles on the same side less than two rt. angles, those two lines on being produced shall intersect, *may very readily be demonstrated by the principles now established.*



Exp.	1	Hyp. 1.	Let AC falling on AB, CD, make the int. ∠s ACD, CAB < two rt. angles;
	2	„ 2.	and let ∠s ACD, CAE = two rt. angles:
	3	Concl. 1.	then AB and CD being produced shall intersect,
	4	„ 2.	and AE and CD shall be parallels.
Cons.	1	Assum. & P. 31	In AB take any point B, and through B draw EF ∥ to AC;
	2	P. 3.	Along EB produced set EB, as many times as will make the distance EB repeated fall below CD; in the present instance EF equalling twice EB, or EB = BF:
	3	P. 31 & 3.	From F draw FG ∥ CD or AE, and = AE;
	4	Concl.	the line ABG is but one st. line.
	5	Def. 35, C.	And ∵ CD cannot cut its ∥ FG, and the line AB concurs with FG;
	6	Concl.	∴ CD shall cut AG between B and G.
Dem.	1	C. 3 & 2, P. 29	In △s AEB, BFG, the side AE = FG, BE = BF, and ∠ AEB = ∠ BFG;
	2	P. 4.	therefore the ∠ EBA = ∠ FBG:
	3	Cor. P. 15.	consequently AB and BG make one st. line, and CD being produced cuts AG in the point H.

Use and Application.—1. The drawing of parallel lines is required in Perspective, Navigation, the Construction of the Sector, or Compass of Proportion, and in other branches of Practical Mathematics.

2. The use of *parallel lines* enables the Surveyor *to ascertain the distance of an inaccessible object*, by the method of Representative Values or of Construction: thus, there are three objects, A, B, C, distant from each other AC 6 miles, AB 8, and BC 9·4 miles; at D, a station on the line BC, the angle CDA is found to be 60°: the distance from D to A is required.



From a scale of equal parts construct the triangle ABC; and at B make the angle CBE equal to 60°; AD, drawn parallel to BE, is the representative value of the distance from D to A. Taking DA in the compasses, and applying the distance to the scale, the measurement of DA will be found equal to 6 miles.

PROP. 32.—THEOR.—*(Very Important.)*

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of every triangle are together equal to two right angles.

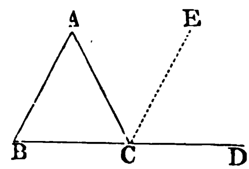
CONSTRUCTION.—P. 31. To draw a st. line, through a given point, parallel to a given st. line.

DEMONSTRATION.—P. 29. A line falling on two parallel lines makes the alternate angles equal, and the exterior angle equal to the int. and opp. angle upon the same side; and likewise the two interior angles on the same side equal to two rt. angles.

Ax. 2. If equals be added to equals, the wholes are equal.

P. 13. The angles which one st. line makes with another upon one side of it, are either two right angles, or together are equal to two rt. angles.

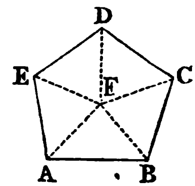
Ax. 1. Magnitudes equal to the same, are equal to each other.



EXP.	1	Hyp.	Let the side BC of the △ ABC be produced to D;
	2	Concl. 1.	then the ext. ∠ ACD = the int. and opp. ∠s CAB, ABC;
		„ 2.	and the three int. ∠s ABC, ACB, and CAB together = two rt. angles.
CONS.	1	P. 31.	Draw from C a line CE ∥ BA.
DEM.	1	C. & P. 29	∵ AB is ∥ to CE, and AC meets them, the ∠BAC = ∠ACE:
	2	C.	Again, ∵ AB is ∥ CE, and BD falls on them,
	3	P. 29.	∴ the ext. ∠ ECD = the int. and opp. ∠ ABC:
	4	D. 1.	But ∠ ACE = ∠BAC;
	5	Ax. 2.	∴ ext. ∠ ACD = the int. and opp. ∠s BAC and ABC:
	6	Add.	To each of these equals add the ∠ ACB;
	7	Ax. 2.	then ∠s ACD + ACB = ∠s BAC + ABC + ACB:
	8	P. 13.	But ∠s ACD + ACB = two rt. angles.
	9	Ax. 1.	∴ ∠s BAC, ABC, ACB together = two rt. angles.
	10	Recap.	*Wherefore, if a side of a triangle be produced,* &c. Q.E.D.

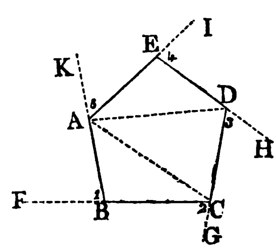
COR. I.—*All the interior angles of any rectilineal figure, together with four rt. angles, are equal to twice as mang rt. angles as the figure has sides.*

DEMONSTRATION.—P. 15, Cor. 2. All the angles made by any number of lines meeting at a point, are together equal to two rt. angles.



EXP.	1	Hyp. 1.	Let ABCDE be a rectil. figure, and F a point within it;
		„ 2.	and let lines be drawn from F to the ang. points A, B, C, D, E;
	2	Concl.	then all the int. ∠s together with four rt. angles = twice as many rt. angles as the figure has sides.
DEM.	1	Hyp.	∵ the fig. ABCDE is divided into as many triangles as there are sides,
	2	P. 32.	and ∵ the ∠s in each △ = two rt. angles;
	3	D. 1 & 2.	∴ all the angles of the triangles = two right angles × the number of sides:
	4	Hyp.	But the same ∠s equal the angles of the fig. + the angles at F;
	5	P15, Cor 2	and all the ∠s at a point, as F, = four rt. angles;
	6	Concl.	∴ the ∠s of the fig. + four rt. angles = two rt. angles × the number of sides. Q.E.D.

COR. II.—*All the exterior angles of any rectilineal figure, as* ABCDE, *are together equal to four rt. angles.*



CONS.	1	Pst. 2.	Produce all the sides of the fig. ABCDE:
DEM.	1	P. 13.	∵ each int. ∠ + the adj. ext. ∠ = two rt. angles;
	2	P32, Cor 1	∴ all the int. ∠s + all the adj. ext. ∠s = two rt. ∠s × the number of sides;
	3	Ax. 1.	and ∴ these = all the int. ∠s + four rt. angles.
	4	„ 2.	∴ the ext. ∠s = four rt. angles. Q.E.D.

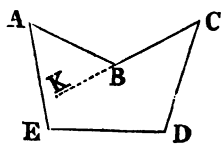
COR. III.—*If two triangles* ABC, ADE *have two angles of the one,* ABC, ACB, *respectively equal to two angles of the other,* AED, ADE; *then the third angle of the one,* BAC, *shall be equal to the third angle of the other,* DAE.

DEM.	1	Hyp.	The $\angle$s ABC + ACB = $\angle$s AED + ADE;
	2	P. 32.	and the $\angle$s ABC + ACB + BAC = two rt. angles;
	3	P. 32.	also the $\angle$s AED + ADE + DAE = two rt. angles:
	4	Sub.	take away the equals ABC + ACB and AED + ADE;
	5	Ax. 3.	$\therefore$ the rem. $\angle$ BAC = the rem. $\angle$ DAE.

Q.E.D.

SCHOLIUM.—1. The Corollaries to be deduced from Prop. 32, are very numerous: Lardner's *Euclid* gives twenty-four.

2. The first Cor. is of universal extent, applying to all rectilineal figures; the second is applicable only to figures that are called *convex,* in which each int. angle is less than two rt. angles. Some figures, as ABCDE, being *concave,* at B, have *re-entrant angles,* which are greater than two right angles: thus the int. angle at B is greater than two rt. angles by the angle ABK.

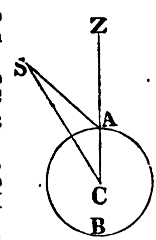


3. Without the production of a side the three angles may be proved to be equal to two rt. angles, by drawing through any angular point a parallel to the opposite side.

4. In an equilateral triangle each angle measures two-thirds of a rt. angle, or 60°; and in an isosceles triangle, if one angle is rt. angled, the other two are each 45°, or half a rt. angle.

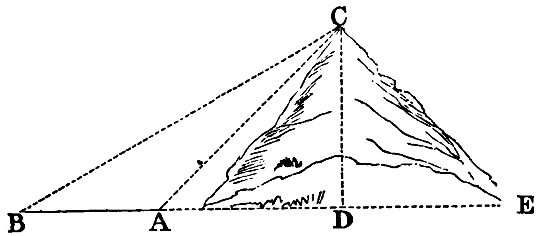
USE AND APPLICATION.—1. This Theorem may be employed in Astronomy *to determine the Parallax of a heavenly body.*

Let C represent the earth's centre, A a point on the earth's circumference AB, and Z the zenith of the station A; S is a star or any heavenly body not in the zenith.



By observation take the angle ZAS, the zenith distance at the earth's surface A: if the star S were viewed from C the centre, the angle ZCS would be its zenith distance; and the angle ZCS is less than angle ZAS. For, by P. 32, the ext. angle ZAS is equal to the angles C + S: thus, the ang. S is equal to the excess of the angle ZAS above the angle ZCS. If now, from an Astronomical Table, I learn what is the zenith distance of S as viewed from C, the earth's centre, the difference between ang. ZAS and ang. ZCS, or the angle ASC will be the parallax.

2. By employing the same Theorem we may construct a figure which will give the representative value of the *perpendicular height of a mountain,* as CD.

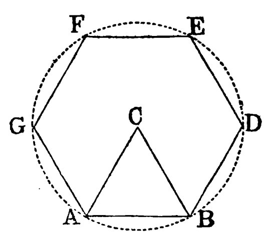


At station A with a theodolite ascertain the angle CAD,—suppose 45°; measure AB, 300 feet, and at station B measure the angle CBD, 30°; and from these data construct the figure required.

From B draw a line of indefinite length BE, and take on it 300 equal parts to A: at B draw an angle CBD of 30°; and at A an angle CAD of 45°: the lines BC and AC intersect in the point C; and a perpendicular from C to BE, namely CD, will represent the perpendicular height of the mountain. Take CD in the compasses, and apply the distance to the same scale;—it will be found that CD measures 400, feet or yards, according to the unit of length in AB.

3. On the principle that all the interior angles of any rectilineal figure are equal to twice as many rt. angles as the figure has sides, diminished by four rt. angles for the amount of the angles at the centre,—we are enabled to construct any regular right-lined figure. The angular magnitude of each figure is equal to two rt. angles multiplied by the number of sides, *minus* four rt. angles; and the remainder divided by the number of sides or angles, gives each single angle of the regular figure. Let S equal the number of sides; 180° the measure of two rt. angles; and 360° the measure of four rt. angles; and let A represent the angle at each pair of sides, and C the angle at the centre by radii to the extremities of each side.

Then $A = \frac{(S \times 180°)-360}{S}$; and $C = \frac{360}{S}$.



To construct a regular Polygon:—Ascertain by the foregoing formula the angle C, and the angle A. 1st. *When the side* AB *is not given*—Draw any radius, as CA, and at C, with another radius of the same length as the first, make the ang. C as ascertained by the formula; then describe with CA or CB the circle, and the distance AB will divide the circumference into as many parts as there are units in S: join the points, and the polygon is formed. 2nd. *When the side* AB *is given*—At A and B make angles equal to ang. A, as ascertained by the formula; and bisect those angles by the lines AC and BC meeting in the point C; triangle CAB having the angles CAB and CBA equal, the sides CA and CB are also equal: if now, with CA or CB as radius, a circle be described, AB will cut off as many arcs from the circumference as there are units in S;—draw the chords to the respective arcs, and the polygon will be completed.

PROP. 33.—THEOR.

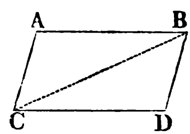
The st. lines which join the extremities of two equal and parallel st. lines towards the same parts, are also equal and parallel.

CONSTRUCTION.—Pst. 1. A line may be drawn from one point to another.

DEMONSTRATION.—P. 29. A line falling on two parallel lines makes the alternate angles equal.

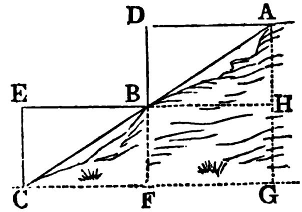
P. 4. Two triangles having in each two sides and their included angle respectively equal, are equal in every other respect.

P. 27. If a line falling on two st. lines make the alternate angles equal, the two lines are parallel.



EXP.	1	Hyp. 1.	Let AB be = and ∥ to CD;
		„ 2.	and let the extremities towards the same parts be joined, A to C, and B to D;
	2	Concl.	then the st. lines AC and BD are = and ∥.
CONS.	1	Pst. 1.	Join B and C.
DEM.	1	H. & P. 29	∵ BC meets the ∥s AB, CD, ∴ the ∠ ABC = the alt. ∠ BCD:
	2	H. & D. 1.	Hence, ∵ AB = CD, BC is common, and ∠ ABC = ∠ BCD,
	3	P. 4.	∴ ∠ ACB = ∠ CBD, and AC = BD.
	4	D. 3.	And ∵ BC with AC, BD, makes ∠ ACB = ∠ CBD,
	5	P. 27.	∴ the st. line AC is ∥ to the st. line BD;
	6	D. 3.	And AC is equal to BD.
	7	Recap.	*Wherefore, the st. lines which join the extremities,* &c. Q.E.D.

SCHOLIUM.—A *Parallelogram* is a four-sided figure of which the opposite sides are equal and parallel, and the diagonals join opposite angles.



USE AND APPLICATION.—The principle contained in this Theorem enables us to *ascertain the perpendicular height of a mountain* AG, *as well as the distance from the base to the foot of the perpendicular,* CG.

Take a large right-angled triangle, ABD, and putting one end at A, let the side DB, by means of a plummet, be brought to the perpendicular; DB shows the altitude AH, and DA the horizontal

distance BH: repeat the process as often as the case requires;—then DA + EB, &c., equals the horizontal distance CG; and DB + EC, &c., the altitude AG.

Prop. 34.—Theor.

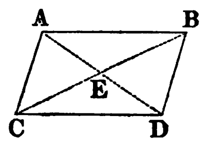
The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them; that is, divides them into two equal parts.

Demonstration.—P. 29. A line falling on two parallel lines makes the alternate angles equal.

P. 26. If two triangles have two angles and a side of one equal to two angles and a side of the other, the other sides and angle are respectively equal.

Ax. 2. If equals be added to equals, the wholes are equal.

P. 4. If two triangles have two sides and the included angle equal in each, they are altogether equal.



Exp.	1	Hyp.	Let ABCD be a ▱, and BC its diameter;
	2	Concl. 1.	then $\angle$ A = $\angle$ D, and $\angle$ B = $\angle$ C;
		" 2.	also $\triangle$ ABC = $\triangle$ BCD.
Dem.	1	H. & P. 29	∵ BC meets the ‖s AB and CD, ∴ the $\angle$ ABC = the alt. $\angle$ BCD;
	2	H. & P. 29	and ∵ BC meets the ‖s AC and BD, ∴ the $\angle$ ACB = the alt. $\angle$ CBD:
	3	D. 1 & 2.	Hence, ∵ $\angle$s ABC, ACB = $\angle$s BCD and CBD, and BC is common to the $\triangle$s ABC, BCD;
	4	P. 26.	∴ $\angle$ BAC = $\angle$ BDC, AB = CD, and AC = BD.
	5	D. 1 & 2.	And ∵ $\angle$ ABC = $\angle$ BCD, and $\angle$ CBD = $\angle$ ACB,
	6	Ax. 2.	∴ the whole $\angle$ ABD = the whole $\angle$ ACD;
	7	D. 4.	and $\angle$ BAC = $\angle$ BDC;
	8	Concl.	∴ the opp. sides and angles of ▱s are equal.
	9	H. & D. 1.	Also, ∵ AB = CD, BC is common, and $\angle$ ABC = $\angle$ BCD;
	10	P. 4.	∴ the $\triangle$ ABC = the $\triangle$ BCD.
	11	Recap.	*Wherefore, the opposite sides and angles of parallelograms, &c.* Q.E.D.

SCHOLIUM.—1. The two diagonals, AD and BC, bisect each other.

2. The converse to Prop. 34 is,—If the opposite sides or opposite angles of a quadrilateral figure be equal, the opposite sides shall be parallel; *i. e.*, the figure shall be a parallelogram.

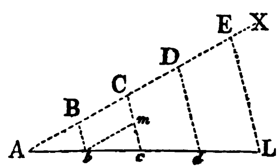
3. Both the diagonals, AD and BC, being drawn, it may, with a few exceptions which require subsequent propositions, be proved, that a quadrilateral figure which has any *two* of the following properties, will also have the others:—

1°. The parallelism of AB and CD;
2°. The parallelism of AC and BD;
3°. The equality of AB and CD;
4°. The equality of AC and BD;
5°. The equality of the angles A and D;
6°. The equality of the angles B and C;
7°. The bisection of AD by BC;
8°. The bisection of BC by AD;
9°. The bisection of the area by AD;
10°. The bisection of the area by BC.

These ten data, being combined in pairs, will give 45 distinct pairs; with each of these pairs it may be required to establish any of the eight other properties, and thus 360 questions, respecting such quadrilaterals, may be raised. These questions will furnish the student a useful geometrical exercise. "The 9th and 10th data require the aid of snbsequent propositions."
—LARDNER'S *Euclid*, p. 49.

USE AND APPLICATION.—1. The construction and accuracy of the parallel ruler depends on Prop. 34.

2. *A finite st. line may be divided into any given number of equal parts.*



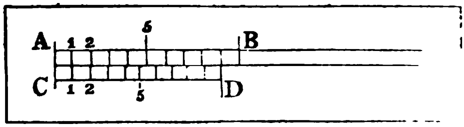
EXP.	1	Data.	Given a line AL, and the number of parts, as four;
	2	Quæs.	to divide AL into those parts.
CONS.	1	Pst. 1.	From one extremity of AL, as A, draw an indefinite line AX,
	2	P. 3 & Pst. 1.	take AB, and make BC, CD, DE each = AB, and join EL;
	3	P. 31.	through D, C, B draw Dd, Cc, Bb, ‖s to EL;
	4	Sol.	the line AL is so divided that Ab=bc, bc=cd, and cd=dL.
	5	P. 31.	Also through b draw bm ∥ AX.
DEM.	1	C. 3 & 5, P. 34	∵ BbCm is a ▱, ∴ bm = BC or AB;
	2	P. 29.	also ∠ A = ∠ cbm, and ∠ AbB = ∠ bcm:
	3	P. 26.	therefore Ab is equal to bc.
	4	Sim.	And in the same way dc = bc, and dL = cd.

Q. E. D.

3. On the same principle the *Sliding Scale*, called from the inventors the *Vernier* or *Nonius*, is constructed. This scale is very useful, as in the

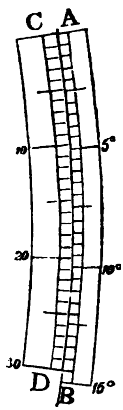
Barometer, for measuring the hundredth part of an inch; or in the Theodolite, for measuring the minutes into which a degree is divided.

1°. *For measuring the hundredth part of an inch*, we may take, because of its clearness, *Ritchie's* description, in his *Geometry*, p. 32. "Suppose an

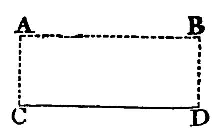


inch AB, divided into 10 equal parts, each part will be $\frac{1}{10}$ of an inch. Again, suppose a line CD equal to 9 of these parts to be also divided into 10 equal parts, each of these will be $\frac{1}{10}$ of $\frac{9}{10}$ of an inch, which is $\frac{9}{100}$. But the length of one of the first divisions being $\frac{1}{10}$, or $\frac{10}{100}$, and that of the second $\frac{9}{100}$, one of the first divisions is $\frac{1}{100}$ of an inch longer than one of the second. If the line CD slide along parallel to AB till the two divisions marked 1, 1, form a continuous line, the sliding scale will have moved $\frac{1}{100}$ part of an inch towards B. If it slide along till the next two divisions coincide, it will have moved $\frac{2}{100}$ of an inch, &c."

2°. *For measuring the minutes into which a degree on a circle is divided.* "If 59 degrees on the circumference or *limb* of the instrument be divided into 60 equal parts, the difference between the length of one degree and one of the latter divisions will obviously be $\frac{1}{60}$ of a degree, or one minute. This kind of Vernier, on account of its great length, is seldom employed. If each degree on the arc AB of 15° be divided into two equal parts or half degrees, and if 29 of these be taken and divided into 30 equal parts, as in CD, the difference between the length of half a degree and one of the new divisions in CD will obviously be $\frac{1}{30}$ of half a degree, or $\frac{1}{60}$ of a degree, that is, one minute. This is a very common vernier." If the limb CD move along the arc AB, so that the first division on CD and the first on AB are in one and the same line, the vernier will have moved $\frac{1}{2}$ of $\frac{1}{30}$, or $\frac{1}{60}$ of a degree, *i. e.*, one minute; if the second division on CD coincide with the second on the arc AB, the vernier will have moved $\frac{2}{60}$ of a degree, or two minutes, &c.

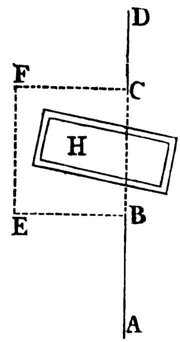


4. *When two objects*, A *and* B, *are inaccessible from each other*, with an instrument to set out perpendiculars from them to a line CD, parallel to the line joining A and B, *the distance may be obtained;* for, by Prop. 34, CD is equal to AB; if therefore CD be measured, we learn the distance from A to B.

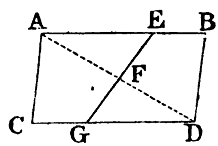


5. Also, *when there intervenes an obstacle*, as a house, or a lake, *to the continuation of a st. line* AB, beyond the obstacle H;—the direction of AB may be ascertained by the method of taking perpendiculars, and of drawing parallel lines.

At B raise a perpendicular BE, and at E another perpendicular EF; at F, a point beyond the obstacle, also draw a perpendicular FC; and set out FC of the same length as BE; then at C draw another perpendicular CD, and the line CD will be in the same direction with the line AB: and were H the obstacle removed, and C and B joined, ABCD would be in one and the same st. line. By construction BEFC is a rectangle, and the angles at C and B rt. angles; therefore, by Prop. 14, AB and CD will be in the same st. line with BC.



6. A field of the shape of *a parallelogram*, ABCD, *may be divided into two equal parts* by the diagonal AD; but if it has to be divided from the point E, bisect the diagonal in F, join EF, and produce EF to G; the line EG will divide the field into two equal portions. For in the triangles AFE, DFG, the angles EAF, AEF are respectively equal to the angles FDG, FGD, by Prop. 29, and AF equal to FD; therefore by Prop. 26, the triangles AFE and FDG are equal. And since the trapezium BEFD and the triangle AFE by Prop. 30 together make up half the field ABD; the same trapezium with the triangle FGD, which is equal to the triangle AFE, will also make up half the field: therefore the line EG divides the field into two equal portions.



Prop. 35.—Theor.

Parallelograms upon the same base and between the same parallels are equal, or rather equivalent, to one another.

Demonstration.—P. 34. The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them.

Ax. 6. Doubles of the same magnitude equal each other.

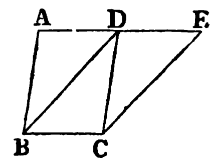
Ax. 1. Things equal to the same are equal to one another.

Ax. 3. If equals are taken from equals, the remainders are equal.

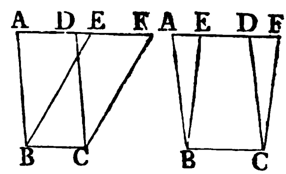
P. 29. A st. line falling on two parallel st. lines makes the exterior angle equal to the int. and opp. angle.

P. 24. Triangles with two sides and their included angle in each equal, are equal in all other respects.

Exp.	1	Hyp. 1.	Let the ▱s ABCD, EBCF, be on the same base BC,
		„ 2.	and between the same ‖s AF and BC;
	2	Concl.	then the ▱ ABCD = the ▱ EBCF.



Case I.		Sup.	*Let the sides* AD, DE *opp. to* BC *be terminated in the same point* D.
Dem.	1	P. 34.	Because ▱s ABCD, DBCE are each double of △ DBC;
	2	Ax. 6.	therefore ▱ ABCD = the ▱ DBCE.
Case II.		Sup.	*Let* AD *and* EF *be not terminated in one point* D,



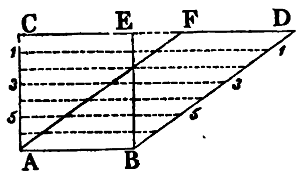
Dem.	1	H. & P. 34.	∵ ABCD is a ▱, ∴ AD = BC;
	2	H. & P. 34.	and EBCF being a ▱, EF = BC;
	3	Ax. 1.	and ∴ AD = EF.
	4	Ad. or Sub.	Add, or take away the common part DE;
	5	Ax. 2 or 3.	∴ the whole, or remainder, AE = the whole, or rem. DF.
	6	P. 34.	And ∵ AE = DF, and AB = DC,
	7	P. 29.	and the ext. ∠ FDC = the int. ∠ EAB;
	8	P. 4.	∴ EB = FC, and △ EAB = △ FDC.
	9	Sub.	From trapezium ABCF take away the equal △s EAB and FDC;
	10	Ax. 3.	the rem. ▱ ABCD = the rem. ▱ EBCF.
	11	Recap.	*Therefore, Parallelograms upon the same base,* &c. Q.E.D.

Scholium.—1. The equality denoted is equality of surface, or area, not equality of sides and angles.

2. Buonaventura Cavalieri, who was a pupil of Galileo, and died A.D. 1647, invented *the Method of Indivisibles*, one of the methods which preceded that of Fluxions. "He considers a line as composed of an infinite number of points; a surface, of an infinite number of lines; and so on."

"This method, absolutely considered, is defective and even erroneous; but the error is of the same kind as that of Leibnitz, who considered a curve as composed of an infinite number of infinitely small chords; and a surface, of infinitely small rectangles. The error in both is one which does not affect the result, for this reason,—that it consists in using the simplifying effect of a certain supposition too early in the process, by which the logic of the investigation may be injured, but the result is not affected."—*Penny Cyclop.* vi., p. 387.

The equality of parallelograms may be explained by this method of Indivisibles, which consists in the supposition that Surfaces are composed of lines, like so many threads in a piece of cloth. Now two pieces of cloth are equal, if in each there be found the same number of threads equal in length, and that the threads are as closely woven in one piece as in the other; and this will be true, though one piece be a rhomboid, and the other a square. For let ABCE represent a square piece of cloth, and ABDF a rhomboidal piece, on the same width AB, and between the same parallels AB, CD. If in the square ABCE as many lines or threads as we please be drawn parallel and equal to AB, as 1, 3, 5, and these lines or threads be produced into and across the rhomboid ABDF, there will be no more lines or threads in the one than in the other; the lines or threads are all of equal length, being equal to the same line or thread AB; and in the one the lines or threads are the same in number, or set as close together, as in the other: consequently, the two pieces of cloth, or the square and rhomboid on the same base AB, and between the same parallels AB and CD, are equal in surface or area.



Use and Application.—In the measurement of Surfaces or Areas, the unit of surface is a rectangle; and it is therefore necessary to convert all parallelograms which are not rectangles into rectangles, in order to find their areas. The following method enables us *to convert the parallelogram* ABDF, *into a rectangle* ABCE, of equal area.

Produce indefinitely the parallel DF, and at B and A, the extremities of the other parallel AB, raise, by Prop. 11, the perpendiculars BE and AC; then ABDF will be converted into a rectangle ABCE, which by Prop. 35, is equivalent to ABDF.

Hence, *the area of a parallelogram is equal to the area of a rectangle having the same base and altitude;* and *if we multiply the linear units in the base by the linear units in the altitude, we obtain the square units, or units of surface, in the parallelogram.*

Prop. 36.—Theor.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Construction.—**Pst. 1. A st. line may be drawn from any one point to any other point.**

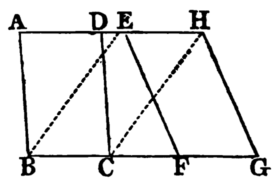
Demonstration.—**P. 34. The opposite sides and angles of parallelograms are equal.**

Ax. 1. Magnitudes equal to the same, are equal to each other.

P. 33. Straight lines joining the extremities of two equal and parallel st. lines towards the same parts, are also themselves equal and parallel.

Def. A. A parallelogram is a four-sided figure of which the opposite sides are parallel; and the diagonal is the st. line joining two of its opposite angles.

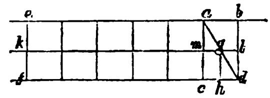
P. 35. Parallelograms on the same base and between the same parallels, are equal.

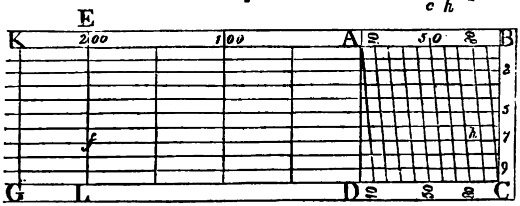


Exp.	1	Hyp. 1.	Let the ▱s ABCD EFGH, be on equal bases, BC and FG,
		„ 2.	and between the same ‖s AH and BG;
	2	Concl.	then ▱ ABCD = ▱ EFGH.
Cons.	1	Pst. 1.	Join the points B, E and C, H.
Dem.	1	H.&P.34.	∵ BC = FG, and FG = EH;
	2	Ax. 1.	∴ BC = EH.
	3	H. & C.	But BC is ‖ EH, and BC and EH are joined towards the same parts by BE, CH;
	4	P. 33.	∴ BE and CH are equal and parallel;
	5	Def. A.	and ∴ also EBCH is a parallelogram.
	6	D.5 & Hyp	Now the ▱s EBCH, ABCD are on the same base BC, and between the same parallels BC, AH;
	7	P. 35.	therefore ▱ EBCH = ▱ ABCD.
	8	P. 35.	Also ▱ EBCH = ▱ EFGH;
	9	Ax. 1.	therefore ▱ ABCD = ▱ EFGH.
	10	Recap.	*Therefore, parallelograms upon equal bases, &c.* Q.E.D.

SCHOLIUM.—The 36th Proposition may be considered as a corollary of the 35th.

USE AND APPLICATION.—1. The *Diagonal Scale* is constructed on the principle of parallelograms on equal bases and between the same parallels being equal: the opposite sides of a parallelogram are divided into the same number of equal parts, and the corresponding parts being joined form similar parallelograms, each one equal to the other; the diagonals to the similar parallelograms are drawn, and thus the Diagonal Scale is constructed.





The principle is thus shown,—"Let *eb*, *kl*, *fd*, be three equi-distant parallel lines, having other equi-distant lines drawn at right angles across them. Join *ad*, then *mg* will be the half of *cd* or *ab*. For draw *gh* at right angles to *cd*, then the triangles *amg*, *ghd* are obviously equal, and hence $mg = hd = ch$; that is, *mg* is the half of *cd*, or its equal *ab*.

"If, instead of drawing one line *kl* between *eb*, *fd*, there be drawn nine equidistant lines," as between the parallels KAB, GDC, "the part *mg* in the second line would obviously be $\frac{1}{10}$ of *ab*, the part on the next line $\frac{2}{10}$ &c."—RITCHIE's *Geom.*, p. 29.

In the Diagonal Scale KBGC thus constructed, the distance AB represents 100, and each of the divisions between A and B 10; and on the diagonal line diverging from A, the first distance from the perpendicular AD to the diagonal will be $\frac{1}{10}$ of 10, or 1; the second distance from AD to the same diagonal, $\frac{2}{10}$ of 10, or 2; the third distance, $\frac{3}{10}$ of 10, or 3; and so on. Thus the spaces between E and A, are *hundreds*; between A and B, *tens*; and between B and C, *units*. If, however, the spaces between E and A are *tens*, those between A and B are *units*, and between B and C *tenths*: indeed the values depend on what we call the spaces between E and A. The extent from *f* on the perpendicular EL, to *h* on the seventh diagonal to the seventh parallel, may be taken for 277, 27·7, or 2·77, according as we consider EA hundreds, or tens, or units.

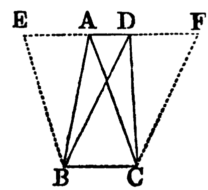
The Diagonal Scale is of very extensive use in the construction and measurement of Geometrical Figures.

PROP. 37.—THEOR.

Triangles upon the same base and between the same parallels are equal to one another.

CONSTRUCTION.—Pst. 2. A st. line may be produced in a st. line.
P. 31. Through a given point to draw a line parallel to a given line.
Def. A. A parallelogram is a four-sided figure of which the opposite sides are parallel, and the diagonal joins opposite angles.

DEMONSTRATION.—P. 35. Parallelograms on the same base and between the same parallels are equal.
P. 34. The opposite sides and angles of parallelograms are equal, and the diameter bisects them.
Ax. 7. Halves of the same magnitude are equal.



EXP.	1	Hyp. 1.	Let the $\triangle$s ABC, DBC be on the same base BC,
		„ 2.	and between the same $\parallel$s AD, BC;
	2	Con. 2.	then the $\triangle$ ABC = the $\triangle$ DBC.
CONS.	1	Pst. 2.	Produce AD both ways indefinitely.
	2	P. 31.	through B draw BE $\parallel$ CA, and through C CF $\parallel$ BD;
	3	Def. A.	then EBCA and DBCF are parallelograms.
DEM.	1	C. 3 & H.	$\because$ the ▱s are on BC, and between the $\parallel$s BC, EF;
	2	P. 35.	$\therefore$ the ▱ EBCA = the ▱ DBCF:
	3	P. 34.	But a ▱ is bisected by its diameter, and $\therefore$ $\triangle$ ABC = half the ▱ EBCA, and $\triangle$ DBC = half the ▱ DBCF;
	4	Ax. 7.	$\therefore$ also $\triangle$ ABC = $\triangle$ DBC.
	5	Recap.	*Wherefore, triangles upon the same base, &c.* Q.E.D.

SCHOLIUM.—On the principle that triangles are the halves of parallelograms, *the areas of triangles* are obtained; the product of the base and altitude gives the area of a parallelogram; consequently, ***half the product of the base and altitude gives the area of the triangle.***

—

Prop. 38.—Theor.

Triangles upon equal bases and between the same parallels, are equal to one another.

Construction.—Pst. 2. A st. line may be produced indefinitely.

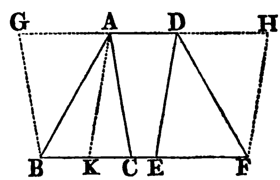
P. 31. To draw a line through a given point parallel to a given line.

Def. A. A parallelogram is a four-sided figure of which the opposite sides are parallel, and the diagonal joins opposite angles.

Demonstration.—P. 36. Parallelograms upon equal bases and between the same parallels are equal.

P. 34. The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them.

Ax. 7. The halves of the same or of equal magnitudes are equal.



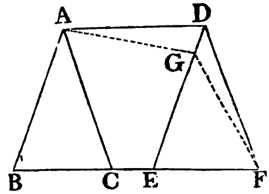
Exp.	1	Hyp. 1.	Let the $\triangle$s ABC, DEF, be on equal bases BC, EF,
		„ 2.	and between the same $\parallel$s BF, AD;
	2	Concl.	then the $\triangle$ ABC = the $\triangle$ DEF.
Cons.	1	Pst. 2.	Produce AD both ways indefinitely;
	2	P. 31.	through B draw BG $\parallel$ CA, and through F, FH $\parallel$ ED;
	3	Def. A.	then GBCA and DEFH are parallelograms.
Dem.	1	H. & C. 1.	$\because$ BC = EF, and BF $\parallel$ GH;
	2	P. 36.	$\therefore$ the $\square$ GBCA = the $\square$ DEFH:
	3	P. 34.	But a parallelogram is bisected by its diagonal; $\therefore$ $\triangle$s ABC, DEF each = half GBCA, DEFH;
	4	Ax. 7.	and $\therefore$ $\triangle$ ABC = $\triangle$ DEF.
	5	Recap.	*Wherefore, triangles upon equal bases, &c.*

Q.E.D.

Scholium.—1. The bases of the triangles are placed so as to form portions of the same st. line.

2. The Area of a triangle may be bisected, or divided into any number of equal parts: in the one case, by bisecting the base; in the other, by dividing the base into equal parts, and joining the points of division and the vertex.

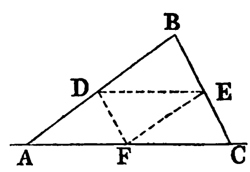
Use and Application.—1. By the last two propositions we arrive at *a practical way of dividing a triangular space, as* ABC, *into two equal parts;* for if the base BC be bisected from the vertex A by AK, then, because the two triangles ABK and ACK are on equal bases BK and CK, and between the same parallels, AD and BC, those triangles are of equal areas.



EXP.	2	Concl.	then AD is parallel to BF.
CONS.	1	Pst. 1.	Join A, D, and AD is parallel to BF.
	2	Sup.	But suppose AD not parallel to BF,
	3	P.31, Pst.1	through A draw AG ∥ BF, and join G, F;
DEM.	1	H. & C. 3.	∵ the △s ABC, GEF are on equal bases, BC, EF, and BF ∥ AG;
	2	P. 38.	∴ the △ ABC = the △ GEF:
	3	Hyp.	But △ ABC also = △ DEF;
	4	Ax. 1.	∴ △ DEF = △ GEF,
	5	*ex abs.*	or, the greater is equal to the less, which is absurd:
	6	Concl.	∴ AG is not parallel to BF.
	7	Sim.	In like manner no line except AD is parallel to BF;
	8	Concl.	therefore AD is parallel to BF.
	9	Recap.	*Wherefore, equal triangles upon the same bases, &c.* Q.E.D.

SCHOLIUM.—1. The point G might be taken in ED produced, and a similar argument pursued.

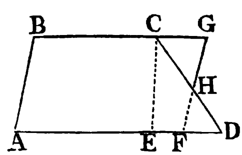
2. From this and the preceding propositions, *Lardner* has deduced sixteen corollaries. Of these the principal are—



1°. A parallel to the base of a triangle through the point of bisection of one side, will bisect the other side.

2°. The lines which join the middle points D, E, F, of the three sides of a triangle divide it into four triangles which are equal in every respect.

3°. The line joining the points of bisection of each pair of sides, is equal to half the third side.



4°. *A trapezium is equal to a parallelogram of the same altitude, and whose base is half the sum of the parallel sides.*

Let CD be bisected in H, and through H a parallel, GHF to AB be drawn.

Since CG and FD are parallel, the angles GCH and G are respectively equal to D, and HFD (29) and CH is equal to HD; therefore (26) CG is equal to FD, and the triangle CHG

to the triangle DHF. Therefore AF and BG are together equal to AD and BC, and the parallelogram AG to the trapezium AC; and since AF and BG are equal, AF is half the sum of AD and BC.

Thus the ***Area of a trapezium*** will be found by taking half the sum of the parallel sides and multiplying that half sum by the altitude CE.

3. "The ***area of a square is found numerically*** by multiplying the number of equal parts in the side of the square by itself. Thus a square whose side is twelve inches, contains in its area 144 square inches. Hence in arithmetic, when a number is multiplied by itself, the product is called its square. Thus, 9, 16, 25, &c., are the *squares* of 3, 4, 5, &c.; and 3, 4, 5, &c., are called the *square roots* of the numbers 9, 16, 25, &c. Thus *square* and *square root* are correlative terms."—LARDNER'S *Euclid*, p. 54.

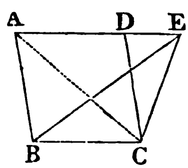
PROP. 41.—THEOR.—*(Important.)*

If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram shall be double of the triangle.

CONSTRUCTION.—Pst. 1. A straight line may be drawn from any one point to any other point.

DEMONSTRATION.—P. 37. Triangles upon the same base and between the same parallels, are equal to one another.

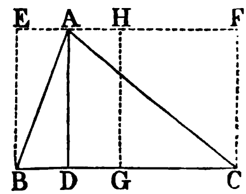
P. 34. The opposite sides and angles of parallelograms are equal to one another; and the diameter, or diagonal, bisects the parallelogram.



EXP.	1	Hyp. 1.	Let the ▱ ABCD, and the $\triangle$ EBC, both be on BC,
		" 2.	and between the same ‖s BC, AE;
	2	Concl.	then the ▱ ABCD shall be double of the $\triangle$ EBC.
CONS.	1	Pst. 1.	Join the points, A, C, by the st. line AC.
DEM.	1	Hyp.	∵ the $\triangle$s ABC, EBC are on the same base BC, and BC ‖ AE;
	2	P. 37.	∴ the $\triangle$ ABC = the $\triangle$ EBC:
	3	P. 34.	But ▱s are bisected by their diagonals;
	4	Concl.	∴ the ▱ ABCD = twice the $\triangle$ ABC;
	5	D. 2.	and ▱ ABCD equals twice the triangle EBC.
	6	Recap.	*Wherefore, if a parallelogram and a triangle,* &c. Q.E.D.

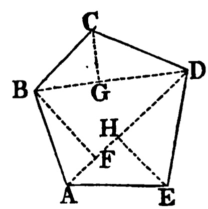
SCHOLIUM.—The converse of Prop. 41 is,—If a parallelogram is double of a triangle, and they have the same base, or equal bases upon the same straight line and towards the same parts; they shall be between the same parallels.

USE AND APPLICATION.—The general method for finding the area of a triangle, or of any figure that may be resolved into triangles, is founded on this proposition. *The area of a parallelogram* is measured by the product of the base and altitude: and a triangle on the same, or on an equal base, being half the parallelogram, *the area of a triangle* is equal to the product of the altitude and half the base, or what is the same thing, equal to half the product of the base and altitude. *The area of polygons* is the sum of the areas of the triangles into which the polygons may be divided.



1. *For the area of the triangle,* ABC:

Take AD perpendicular to BC; bisect BC by GH also perpendicular to BC;
and complete the ▱s ADBE, ADCF.
The area of the triangle ABC equals AD × BG, or AD × $\frac{1}{2}$ BC.

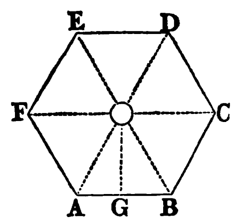


2. *For the area of any figure made up of straight lines, as* ABCDE:

Since the whole figure ABCDE is made up of the triangles ABD, BCD, and DEA;

The area of ABCDE equals the sum of the areas of the triangles ABD, BCD, and DEA:

That is, the area of ABCDE equals (CG × $\frac{1}{2}$ BD) + (BF × $\frac{1}{2}$ AD) + (EH × $\frac{1}{2}$ AD).



3. *For the area of a regular polygon, as* ABCDEF:

From the centre O divide the polygon into triangles, and drop the perpendicular OG to AB;

Then the area of triangle AOB equals OG × $\frac{1}{2}$ AB;

And the area of ABCDEF equals (OG × $\frac{1}{2}$ AB, or AG) × the number of triangles in the figure; or equals the perp. OG × $\frac{1}{2}$ the perimeter.

4. Dividing a circle by an infinite number of triangles having their common vertex in the centre, *the area of a circle equals* the product of the radius and of the semi-circumference. This method depends on the principle that the circle is the limit of the polygons inscribed in it; the circumference is the limit of their perimeters; and the radius the limit of their perpendiculars: and as the area of any polygon equals the perpendicular

multiplied by half the perimeter, the area of the circle, therefore, is equal to the radius × $\frac{1}{2}$ the circumference.

It was ARCHIMEDES, of Syracuse, the most famous of ancient mathematicians (born B.C. 287) who first demonstrated this principle, and established the ratio between the diameter and the circumference of a circle. The circumference of a circle he showed "to be greater than three times its diameter, by a quantity greater than $\frac{10}{71}$ of the diameter, but less than $\frac{1}{7}$ of the same." The proportion now employed is as 1 to 3·14159265, &c., or for general purposes, as 1 to 3·1416: that is, when the diameter measures 1 foot, or yard, or mile, the circumference will measure 3·1416 feet, or yards, or miles.

If we take D to represent the diameter, C the circumference, A the area of a circle, and $p = 3{\cdot}1416$; by having any two of these given, we can ascertain the others: thus,

$$1^{\circ}.\ D=\frac{C}{p}=\frac{4A}{C}=2\sqrt{\frac{A}{p}}. \qquad 3^{\circ}.\ A=\frac{pD^2}{4}=\frac{C^2}{4p}=\frac{DC}{4}.$$

$$2^{\circ}.\ C=pD=\frac{4A}{D}=2\sqrt{pA}. \qquad 4^{\circ}.\ p=\frac{C}{D}=\frac{4A}{D^2}=\frac{C^2}{4A}.$$

PROP. 42.—PROB.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

SOLUTION.—P. 10. To bisect a finite st. line.

Pst. 1. A st. line may be drawn from one point to another.

P. 23. At a point in a st. line to make an angle equal to a given angle.

P. 31. To draw through a given point a parallel to a given line.

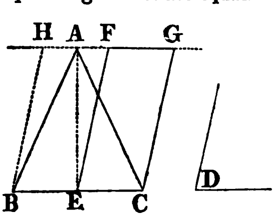
Def. A. A parallelogram is a four-sided figure, of which the opposite sides are parallel.

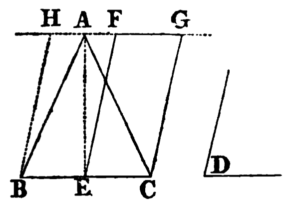
DEMONSTRATION.—P. 38. Triangles upon equal bases and between the same parallels are equal to one another.

P. 41. If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram shall be double of the triangle.

Ax. 6. The doubles of the same or of equal magnitudes are equal.

EXP.	1	Data.	Given the △ABC and the ∠D;
	2	Quæs. 1.	to desc. a ▱ = the △ABC,
		" 2.	and having an angle = the ∠D.





CONS.	1	P. 10, Pst. 1	Bisect BC in E, and join AE;
	2	P. 23.	at E in EC make the ∠ CEF = the ∠ D;
	3	P. 31.	and at A draw AFG ∥ BC, and at C, CG ∥ EF;
	4	Def. A.	then EFGC is a parallelogram,
	5	Sol.	and equal to the triangle ABC, and having an angle = ∠ D.
DEM.	1	C. 1 & 3.	∵ BE = EC, and BC ∥ AG;
	2	P. 38.	∴ △ ABE = △ AEC, and △ ABC = twice △ AEC:
	3	C.	But the ▱ FECG and the △ AEC are both on EC, and EC is ∥ AG;
	4	P. 41.	∴ the ▱ FECG = twice the △ AEC;
	5	Ax. 5.	and ∴ the ▱ FECG = the △ ABC.
	6	C. 2.	The ∠ CEF of ▱ FECG = the ∠ D.
	7	Recap.	*Wherefore, there has been described a parallelogram, &c.* Q.E.F.

SCHOLIUM.—*To describe a triangle equal to a given parallelogram,* FECG, *and having an angle equal to a given angle,* D. As in the last figure, produce the parallels GF and CE, and make EB equal to EC; at B construct an angle equal to the given angle D, and produce BA to meet the parallel GA in A, and join AC; then the triangle ABC, thus constructed will, on drawing BH parallel to FE, be found by Prop. 41 equal to the parallelogram FECG.

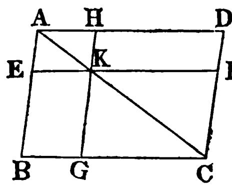
PROP. 43.—THEOR.

The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

DEMONSTRATION.—P. 34. The opposite sides and angles of parallelograms are equal, and the diameter bisects them.

Ax. 2. If equals be added to equals, the wholes are equal.

Ax. 3. If equals be taken from equals, the remainders are equal.



Exp.	1	Hyp. 1.	Let ABCD be a ▱, & AC its diameter,
		" 2.	EH and GF ▱s about AC,
		" 3.	and BK and KD the ▱s complements of the figure;
	2	Concl.	then the complement BK = the complement KD.
Dem.	1	Hyp. 1.	∵ ABCD is a ▱ and AC its diameter;
	2	P. 34.	∴ the △ ABC = the △ ADC:
	3	Hyp. 2.	And ∵ AEKH is a ▱, and AK its diameter,
	4	P. 34.	∴ the △ AEK = the △ AHK:
	5	P. 34.	Also the △ KGC = the △ KFC.
	6	D. 4 & 5.	Then, ∵ △ AEK = △ AHK, and △ KGC = △ KFC;
	7	Ax. 2.	the △s AEK and KGC = the △s AHK and KFC:
	8	D. 2.	But the whole △ ABC = the whole △ ADC;
	9	Ax. 3.	therefore the rem. complement BK = the rem. complement KD.
	10	Recap.	*Wherefore, the complements of the parallelograms,* &c. Q.E.D.

Scholium.—The parallelograms about the diagonal, and also their complements, have each an angle in common with the whole parallelogram, and therefore are equiangular with it.

Use and Application.—If any parallelogram, as KD in the last figure, be given, *another parallelogram*, KB, may be found, *equal to it*, and *having one side*, EK, *equal to a given line*. For, produce FK, and from K set off a distance equal to the given line EK; produce the sides DH, DF, and HK indefinitely; and through E draw AB parallel to HG or DC; draw the diagonal AK, and produce it until it cuts DF produced in C; through C draw a parallel to KF or HD, and it will cut HK and AE produced in the points G and B:—then the parallelogram BK will be equal to the given parallelogram KD.

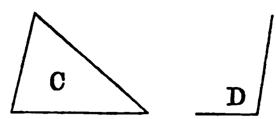
Prop. 44.—Prob.

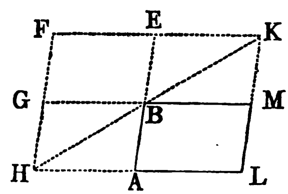
To a given line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given angle.

Solution.—P. 42. To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

P. 31. To draw a line, through a point, parallel to a given line.
Pst. 1. A line may be drawn between any two points.
Pst. 2. A terminated st. line may be produced to any length in a st. line.

Demonstration.—P. 29. If a line fall upon two parallel lines, it makes the alternate angles equal, and the ext. angle equal to the int. and opposite angles, and likewise the two interior angles upon the same side equal to two rt. angles.
Ax. 9. The whole is greater than its part.
Ax. 12. A st. line meeting two st. lines, so as to make the two int. angles on the same side less than two rt. angles, these two lines, being produced, shall meet on the side on which the angles are less than two rt. angles.
P. 43. The complements of the parallelograms which are about the diameter of any parallelogram are equal.
Ax. 1. Magnitudes equal to the same are equal to one another.
P. 15. If two st. lines cut one another, the vertical angles are equal.





Exp.	1	Data.	Given the line AB, △ C, and ∠ D;
	2	Quæs.	to apply to AB a ▱ = the △ C, and having an ang. = the ∠ D.
Cons.	1	P. 42.	Make a ▱ BEFG = the △ C, and having an ang. EBG = ∠ D;
	2	Pos.	place the ▱ so that the side BE form one st. line with AB;
	3	Pst. 2.	produce FG to H;
	4	P.31, Pst.1	and through A draw AH ∥ BG or EF, and join HB:
	5	Pst. 2.	Next produce HB and FE to meet in the point K;
	6	P.31, Pst.2	through K draw KL ∥ AE; and produce GB and HA to M and L;
	7	Sol.	then the ▱ BL = the △ C, and has its ang. ABM = ∠ D.
Dem.	1	C. 3 & 4.	∵ HF falls on the parallels AH and EF,
	2	P. 29.	∴ the ∠ s AHF and HFE together = two rt. angles;
	3	Ax. 9.	and ∠ s BHF & HFE together < two rt. ∠ s;

DEM.	4	Ax. 12.	but when int. angles on the same side are < two rt. angles, their sides meet, if produced;
	5	Concl.	therefore HB and FE being produced will meet:
	6	Pst. 2.	Let them be produced & meet in the point K;
	7	P. 31.	through K draw KL ∥ EA or FH;
	8	Pst. 2.	and produce HA to L, and GB to M;
	9	Concl.	then FKLH is a ▱, HK the diameter, AG and ME ▱s about HK, and LB and FB the complements;
	10	P. 43.	and ∴ compl. LB = compl. BF:
	11	C. 1 & Ax. 1	But ▱ BF = △ C, ∴ ▱ LB = △ C.
	12	P. 15 & C. 1	And ∵ ∠ GBE = ∠ ABM, and ∠ GBE = ∠ D,
	13	Ax. 1.	∴ also ∠ ABM = ∠ D.
	14	Recap.	*Therefore, to the st. line* AB *the parallelogram,* &c. Q.E.F.

SCHOLIUM.—1. In this Problem the Solution and the Demonstration are almost unavoidably mixed together.

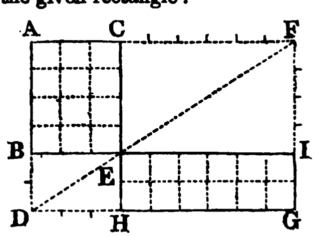
2. When a parallelogram is drawn on a straight line, it is said to be applied to that line.

USE AND APPLICATION.—This proposition contains a kind of *Geometrical Division*. The whole area of one figure, as C, or BF, being given, and AB the side of another; what is required is, so to separate the parts of the given figure, that when applied to the given line, the same number of parts shall be contained in the required figure, as existed in the given figure.

Arithmetically we can say, a given triangle contains, for example, 20 square feet,—a given line 4 lineal feet; how can we apply a parallelogram to the 4 lineal feet, so that the area of the parallelogram shall equal that of the triangle? We divide 20 by 4, and the quotient 5 is the lineal measurement of the other side of the parallelogram.

Geometrically we say, a given rectangle BC contains 12 square feet,—a given line, BD, or its equal EH, 2 lineal feet; how can we apply another rectangle to BD or EH measuring 2 lineal feet, so that the area of the required rectangle may equal the area of the given rectangle?

Produce indefinitely the parallels AC and BE, AB and CE;
from B set off BD equal to the representative value of 2 feet, and draw the diag. DE;
produce DE until it cuts AC produced in F, the diagonal DF will so divide or intersect AC produced, that the part CF shall equal the other side of the rectangle, of which the given side BD is 2 lineal feet.



For, complete the rectangle ADGF: EH equals BD, and EI equals CF; and the complement EG is equal to the complement EA: and if we divide AE into equal squares, and EG into equal squares, we find that in the first there are 12, and in the second 2×6, or 12 also.

—

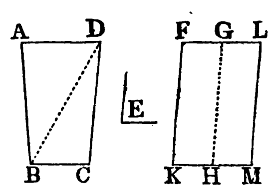
Prop. 45.—Prob.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Solution.—Pst. 1. A line may be drawn to join any two points.
P. 42. To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.
P. 44. To a given line to apply a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.

Demonstration.—Ax. 1. Magnitudes which are equal to the same, are equal to each other.
Ax. 2. If equals be added to equals, the wholes are equal.
P. 29. If a line falls on two parallel lines it makes the alternate angles equal; and the ext. angle equal to the int. and opp. angle on the same side; and the two int. angles on the same side equal to two rt. angles.
P. 14. If at a point in a st. line two other lines on opposite sides make the adj. angles equal to two rt. angles, the two lines are in one and the same st. line.
P. 30. St. lines parallel to the same line are parallel to each other.
Def. A. A parallelogram is a four-sided figure of which the opposite sides are equal and parallel, and the diagonals join opposite angles.

Exp.	1	Data.	Given a rectilin. fig. ABCD, and an $\angle$ E;
	2	Quæs.	to describe a ▱ = ABCD, having an angle = $\angle$ E.
Cons.	1	Pst. 1.	To divide the fig. into triangles, join DB;
	2	P. 42.	describe a ▱ FH = the $\triangle$ ADB, and having an $\angle$ FKH = $\angle$ E;
	3	P. 44.	to the line GH apply the ▱ GM = the $\triangle$DBC, and having an ang. = $\angle$ E;
	4	Sol.	then the fig. FKML is the ▱ required.

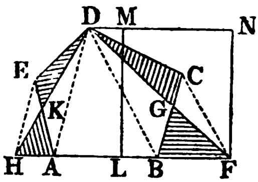


DEM.	1	C. 2 & 3.	∵ ∠ E = ∠ FKH, and also ∠ GHM;
	2	Ax. 1.	∴ ∠ FKH = ∠ GHM:
	3	Add.	To each of the equal angles add the ∠ KHG;
	4	Ax. 2.	then the ∠ s FKH + KHG = ∠ sGHM + KHG:
	5	P. 29.	But ∠ s FKH and KHG = two rt. angles;
	6	Ax. 1.	∴ ∠ s GHM and KHG = two rt. angles:
	7	D. 6.	Thus at H in HG, the adj. ∠ s GHM, GHK = two rt. angles;
	8	P. 14.	∴ KH is in the same st. line with HM.
	9	C.	Again, ∵ HG meets the parallels KM and FG,
	10	P. 29.	∴ ∠ MHG = ∠ HGF;
	11	Add.	To each of the equal angles add ∠ HGL;
	12	Ax. 2.	then ∠ s MHG and HGL = ∠ s HGF and HGL:
	13	P. 29.	But ∠ s MHG and HGL = two rt. angles;
	14	Ax. 1.	∴ ∠ s HGF and HGL = two rt. angles.
	15	P. 14.	And ∵ FG is in the same st. line with GL,
	16	C. 2 & 3.	and KF ∥ HG, and HG ∥ ML;
	17	P. 30.	∴ KF is parallel to ML:
	18	C. 2 & 3.	Also KM is parallel to FL;
	19	Def. A.	∴ KFLM is a parallelogram.
	20	C. 2 & 3.	And ∵ △ ABD = ▱ HF, and △ BDC = ▱ GM;
	21	Ax. 2.	therefore the whole fig. ABCD = the whole ▱ KFLM.
	22	Recap.	*Wherefore, the parallelogram* KFLM *has been described*, &c. Q.E.F.

COR.—Hence a parallelogram equal to a given rectilineal figure can be applied to a given right line and in a given angle, by applying to the given line a parallelogram equal to the first triangle.

USE AND APPLICATION.—1. By this and the preceding Problem we may measure the superficial content of any rectilineal figure whatever, by first reducing it to triangles, and then making a rt. angled parallelogram equal to the sum of the triangles. We may also make a rt. angled parallelogram on a given line, and which shall be equal in area to several irregular figures. Also if we have several figures, we may make another equal to their difference.

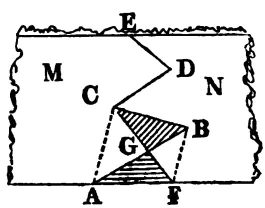
2. By principles already established, and by the Problems for the conversion of rectilineal figures into parallelograms of equal area, *we may change any right-lined figure*, as ABCDE, *first into a triangle, and then into a rectangle of equal area.*



Join DA and DB to divide the given figure into triangles, and produce AB indefinitely. Through E draw EH parallel to DA, and through C, CF parallel to DB: join DH and DF;—then the triangle DHF is equal in area to the given figure ABCDE.

Next, through D draw DN parallel to HF; bisect HF, a side of the triangle DHF, in L; at L raise the perpendicular LM, and through F draw FN a parallel to LM; the fig. LMNF is a rectangle of the same altitude as the triangle, and on half its base,—and is therefore equal in area to the triangle DHF.

3. In a similar way *a crooked boundary*, ABCDE, between two fields, M and N, *may be made straight* without changing the relative size of the fields.



Draw AC the subtend to angle B, and through B, a st. line BF parallel to AC, and join CF; the crooked boundary AB, BC is now converted into the single boundary CF. In a similar way FC and CD will be converted into one boundary, and this last and DE into a single boundary; and thus the crooked boundaries AB, BC, CD, and DE, will be changed into one straight boundary without affecting the size of the fields; the shape will be altered but the areas of M and N remain as at first.

Prop. 46.—Prob.

To describe a square on a given straight line.

Solution.—P. 11. To draw a st. line at right angles to a given line from a given point in the same.

P. 3. From the greater of two lines to cut off a part equal to the less.

P. 31. To draw a line through a point parallel to a given line.

Def. A. A parallelogram is a four-sided figure of which the opposite sides are equal and parallel, and the diagonals join opposite angles.

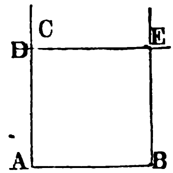
Demonstration.—P. 34. The opposite sides and angles of parallelograms are equal.

Ax. 1. Things equal to the same are equal to each other.

P. 29. If a line fall on two parallel st. lines, it makes the alternate angles equal to each other; and the ext. angle equal to the int. and opposite angle on the same side; and the two int. angles equal to two rt. angles.

Ax. 3. If equals be taken from equals, the remainders are equal.

Def. 30. Of four-sided figures a square is that which has all its sides equal, and all its angles rt. angles.



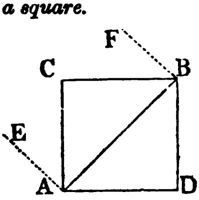
EXP.	1	Datum.	Given the st. line AB;
	2	Quæs.	it is required to draw a square on AB.
CONS.	1	P. 11.	From A draw AC at rt. angles to AB;
	2	P. 3.	make AD = AB;
	3	P. 31.	through D draw DE ∥ AB, and through B, BE ∥ AD;
	4	Def. A.	and ABDE is a parallelogram,
	5	Sol.	and the square required.
DEM.	1	P. 34 & C. 2	∵ AB = DE, AD = BE, and AB = AD;
	2	Ax. 1.	∴ AB = AD = DE = EB;
	3	D. 1 & 2.	and ∴ the ▱ ABDE is equilateral.
	4	C. 3.	Also, ∵ AD meets the parallels AB and DE,
	5	P. 29.	∴ the int. ∠s BAD, ADE = two rt. angles:
	6	C. 1.	but the ∠ BAD is a rt. angle;
	7	Ax. 3.	∴ ∠ ADE is also a rt. angle.
	8	P. 34.	Now the opposite angles of ▱s are equal;
	9	D. 7 & 6.	∴ ∠ ABE opposite to ∠ ADE is a rt. angle, and ∠ BED opposite to ∠ BAD is a rt. angle;
	10	D. 6, 7, & 9.	∴ the figure ADEB is a rectangle;
	11	D. 3.	it is also equilateral;
	12	Def. 30.	therefore □ ABED is a square on AB.

Q.E.F.

COR. 1.—*The squares on equal lines are equal; and if the squares are equal, the lines are equal.*

2. *Every parallelogram having one right angle, has all its angles rt. angles.*

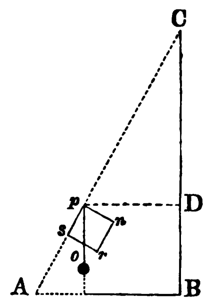
SCHOLIUM.—*Given the diagonal* AB, *to construct a square.*

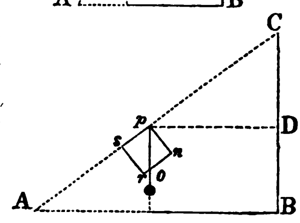


CONS.	1	P. 11.	At A and B draw perpendiculars AE and BF;
	2	P. 9.	bisect the rt. angles by AC and BC meeting in C;
	3	P. 31.	through B and A draw BD ∥ AC, and AD ∥ CB;
	4	Sol.	then ABCD is the square required.
DEM.	1	C. 2, & P. 6.	∵ the ∠s CAB and CBA are equal, AC = CB;
	2	P. 34.	and ∵ ACBD is a ▱, AD = CB, and BD = CA;
	3	Concl.	∴ AC = CB = BD = DA, and the fig. is equilateral.
	4	C.	Again, the ∠s CAB, CBA being together one rt. angle,
	5	P. 32.	the angle C is a rt. angle;

DEM.	6	P. 46, Cor. 2.	But in a ▱, as ACBD, when one angle is a rt. angle, all the angles are rt. angles;
	7	C. 3. Concl.	∴ also ACBD has its angles rt. angles;
	8	Concl.	And therefore, the figure being equilateral and rectangular, ACBD is the square required.

USE AND APPLICATION.—The *Geometrical Square* is an instrument *by means of which*, and of the property of similar triangles that the sides about the equal angles are proportional, *the height of an inaccessible object can be ascertained*, provided a measurement to the perpendicular from the object can be made. The edges of the square are each divided into 100 equal parts, and from one corner a plummet is suspended; when the object is seen along one edge of the instrument, the plummet cuts another edge, and forms a triangle similar to the triangle formed by lines representing the perpendicular from the object, a parallel to the horizontal line at its base, and the hypotenuse, or distance from the point of observation, to the object itself.





In the adjoining figures, AB represents the horizon; pD a parallel to the horizon; DB the height of the instrument; CD the height of the object C above the parallel to the horizon; sp the edge along which the object is to be seen; sr, rn, and pn graduated edges each of 100 parts; and p the point of suspension for the plummet.

From the place of observation measure the distance pD, and the height of the instrument DB; direct the edge sp towards the object C, and note the number of parts in sr or in rn. 1°. When the plummet cuts sr in o, the triangle pso is similar to the triangle CDp; and we have the proportion $so : sp :: pD : CD$; whence $CD = \frac{sp \cdot pD}{so}$ and $CB = CD + DB$. 2°. When the plummet cuts rn in o, the triangles onp and CDp are similar; and we have the proportion $pn : no :: pD : DC$; whence $CD = \frac{no \cdot pD}{pn}$; and $CB = CD + DB$.

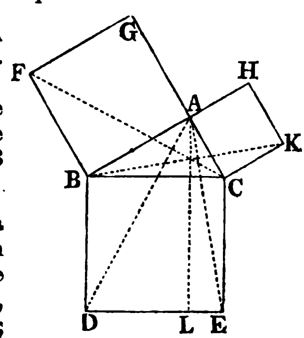
For example, let $pD = 60$ft.; $so = 50$ eq. pts.; and $DB = 6$ ft.; required CB. Here $50 : 100 :: 60 : 120 = CD$, and $120 + 6 = 126$ft. $= CB$. —*See* TATE's *Geometry*, pp. 49–51.

PROP. 47.—THEOR.—*(Most Important.)*

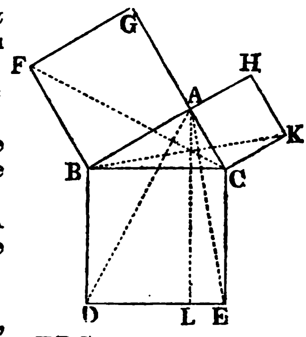
In any right-angled triangle, the square which is described upon the side subtending, or opposite to, the right angle, is equal to the squares described upon the sides containing the right angle.

Construction.—P. 46. To describe a square on a given straight line.
P. 31. Through a point to draw a line parallel to a given line.
Pst. 1. Any two points may be joined by a st. line.

Demonstration.—Def. 30. A square has all its sides equal, and its angles rt. angles.
P. 14. If at a point in a st. line two other st. lines on the opposite sides of it make the adjacent angles together equal to two rt. angles, these two lines shall be in one and the same st. line.
Ax. 1. Things equal to the same are equal to each other.
Ax. 2. If equals be added to equals, the wholes are equal.
P. 4. If two triangles have each two sides and their included angle equal, the triangles are in every other respect equal.
P. 41. If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram is double of the triangle.
Ax. 6. Things double of the same are equal to each other.



Exp.	1	Hyp.	Let ABC be a △ and BAC a rt. angle;
	2	Concl.	then the square on BC = the squares on AB and AC.
Cons.	1	P. 46.	Draw on BC a square BE; on BA, a square BG; & on AC, a square HC;
	2	P. 31.	through A draw AL ∥ BD or to CE;
	3	Pst. 1.	join AD and FC; also AE and BK.
Dem.	1	H.& Def. 30	∵ ∠s BAC and BAG are each a rt. angle;
	2	P. 14.	∴ the lines AC, AG on opp. sides of AB make the adj. ∠s = two rt. angles, and ∴ CA is in the same st. line with AG.
	3	H. Def. 30 & P. 14.	Also AB is on the same st. line with AH.



DEM.	4	C.1, Def.30	∵ the ∠s DBC & FBA are each a rt. angle;
	5	Ax. 1.	∴ the ∠ DBC = the ∠ FBA:
	6	Add.	To each of the equals add the ∠ABC;
	7	Ax. 2.	∴ the ∠ DBA = the whole ∠ FBC:
	8	C.1, Def.30	Hence, ∵ AB = FB, BD = BC, and ∠ DBA = ∠FBC;
	9	P. 4.	∴ AD = FC, and △ ABD = △ FBC.
	10	C. 2.	Now the ▱ BL and the △ ABD are both on BD, and between the same parallels BD and AL;
	11	P. 41.	∴ the ▱ BL is double of the △ ABD.
	12	C. 2.	Also the square GB and the △ FBC are both on FB, and betw. the same parallels FB and GC;
	13	P. 41.	∴ the ▱ or square GB, is double of the △FBC:
	14	D. 9.	But the △ ABD = the △ FBC;
	15	Ax. 6.	therefore the ▱ BL = the square GB.
	16	C. 3.	Also, after joining AE, BK, the ▱ CL = the square HC;
	17	Ax. 2.	therefore the whole square BDEC = the two squares GB and HC.
	18	C. 1.	Now the squares are BE on BC, BG on BA, and CH on CA;
	19	Concl.	therefore the square on BC = the two squares on BA and CA.
	20	Recap.	*Therefore, in any right-angled triangle*, &c.

Q.E.D.

COR. 1.—"Hence, *if the sides of a rt. angled triangle be given in numbers, its hypotenuse may be found:* for let the squares of the sides be added together, and the square root of their sum will be the hypotenuse." Suppose AB the base, AC the perpendicular, BC the hypotenuse; the formula for BC is,

$$BC = \sqrt{AB^2 + AC^2}.$$

Example. The height of a tower is 40 feet, and the breadth of its ditch 30 feet: required the length of a ladder to reach from the further side of the ditch to the top of the tower.

$\sqrt{40^2+30^2}$ or $\sqrt{1600+900}$, or $\sqrt{2500} = 50$ ft., length of the ladder.

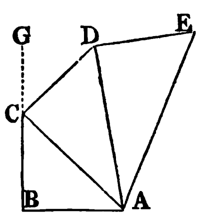
COR. 2.—*If the hypotenuse and one side be given in numbers, the other side may be found:* for let the square of the side be subtracted from that of the hypotenuse, and the remainder is equal to the square of the other side. The square root of this remainder will therefore be equal to the other side. Thus,

$$AB = \sqrt{BC^2 - AC^2}; \text{ and } AC = \sqrt{BC^2 - AB^2}.$$

Example. One village A is at right angles with two others B and C; the distance from B to C is 50 furlongs; from A to C 30 furlongs: required the distance from A to B.

$\sqrt{50^2-30^2} = \sqrt{2500-900} = \sqrt{1600} = 40$ furlongs from A to B.

COR. 3.—*If any number of squares be given, a square equal to their sum may be found; or if one square be given, any multiple of it may be ascertained; or if two squares be given, the difference between them; or a square may be made that shall be the half, fourth, &c., of a given square.*

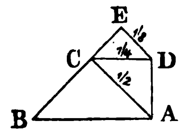


1°. Set the lines AB and BC, representative of the sides of the first two given squares, at rt. angles ABC,—the square on $AC = AB^2 + BC^2$; at C place CD, representative of the third square, at rt. angles to AC,—the square on $AD = AB^2 + BC^2 + CD^2$; and at D place ED, representative of the fourth square, at rt. angles to AD,—the square on $AE = AB^2 + BC^2 + CD^2 + DE^2$.

2°. Supposing AB to be representative of the line on which the given square is constructed, its multiple square will be obtained in a similar way; for in this case BC, CD, DE being each equal to AB, the square on AE is the multiple of the square on AB.

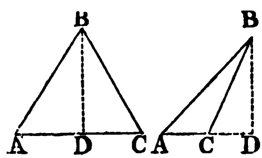
3°. Let AB be the less, and AC the greater of two lines; at B the extremity of the less, raise a perpendicular BG, and from A at the other extremity with AC as radius, inflect on BG the greater line; the square of the intercept CB will equal the difference of the squares on AC and AB.

4°. Make the angles A and B each equal to half a right angle; C being a rt. angle, the square on AC will be one half of the square on AB: again, at A and C make the angles CAD, ACD each equal to half a rt. angle, and the square on CD will be $\frac{1}{2}$ of that on AC, or $\frac{1}{4}$ of the square on AB: the process may be continued for any bisection of the rt. angles supposed to be formed at the extremities of a line, as for $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c.



Cor. 4.—"*If a perpendicular* BD *be drawn from the vertex of a triangle to the base, the difference of the squares of the sides* AB *and* CB, *is equal to the difference between the squares of the segments* AD *and* DC."

For (47. I.) $AB^2 = AD^2 + BD^2$;
and $CB^2 = DC^2 + BD^2$.
Take the latter from the former,
$AB^2 - CB^2 = AD^2 - DC^2$.
The difference vanishes when
$AD = DC$.

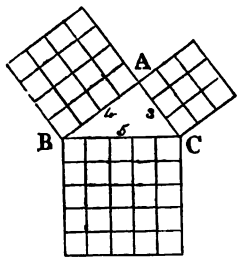


Cor. 5.—"*If a perpendicular be drawn from the vertex* B *to the base* AC, *or* AC *produced, the sums of the squares of the sides and alternate segments are equal.*"

For $AB^2 + BC^2 = AB^2 + BD^2 + DC^2$; and $AB^2 + BC^2 = BC^2 + AD^2 + BD^2$;
therefore (Ax. 1) $AB^2 + BD^2 + DC^2 = BC^2 + AD^2 + BD^2$.
Take away the common square BD^2;
and (Ax. 3) $AB^2 + CD^2 = BC^2 + AD^2$.

Scholium.—1. The 32nd and 47th Propositions are said to have been discovered by Pythagoras, born B.C. 570, and other principles of Geometry to have been brought by him from Egypt into Greece. Whatever we may think of the tale of his extravagant joy on the discovery of the 47th, certain it is that this is one of the most important propositions in all Euclid, for on it, and on the proposition in the sixth book which establishes the similitude of equiangular triangles, the whole science of Trigonometry is founded.

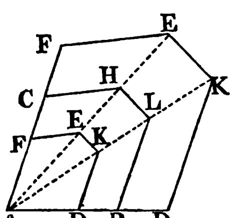
2. A plain and practical illustration of the 47th Proposition may be given by taking three lines in the proportion of 3, 4, and 5, and constructing with them a rt. angled triangle BAC: on each of the three sides draw a square, and sub-divide each square; that on AC 3, into nine smaller squares, that on AB 4, into sixteen, and that on BC 5, into twenty-five squares: the sum of the squares, 9 + 16, on AC and AB, equals the squares on BC.



To form a rt. angle, lines containing 3, 4, and 5 equal parts, or any equimultiples of them, may be used: thus, measure off a line containing 5 feet or links, &c.; at one extremity B, with 4 feet or links, draw an arc, and at the other extremity C, with 3 feet or links, another arc: the two arcs intersect in A, and the lines from A to C, and from A to B, are at right angles to each other.

Use and Application.—1. The 47th and its corollaries may be applied for the construction of all similar rectilineal figures by Prop. 31, bk. vi., where it is proved "that in right-angled triangles, the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly-described figures upon the sides containing the right angle;" and to the making of a circle the double or the half of another circle, by Prop. 2, bk. xii., "that circles are to one another as the squares of their diameters."

1°. *To make a rectilineal figure* ADKEF, *similar to a given rectilineal figure* ABLHC.



Divide the figure into triangles by the lines AH, AL produced, if necessary; take AD equal to the side of the required figure, and through D draw DK parallel to BL, through K, KE parallel to LH, and through E, EF parallel to HC. Then ADKEF will be similar to ABLHC,—for by Prop. 29, the angles D, K, E, F are equal to those at B, L, H, and C, and the triangles ADK similar to ABL, AKE to ALH, and AEF to AHC.

Now, by Prop. 19, bk. vi., "Similar triangles are to each other as the squares of their like sides;" and figures made up of similar triangles being similar, all similar figures are to each other as the squares of their like sides.

For, $\frac{\text{area AEF}}{\text{area AHC}} = \frac{AE^2}{AH^2} = \frac{AK^2}{AL^2} = \frac{AD^2}{AB^2} \therefore \frac{\text{area AEF}}{AD^2} = \frac{\text{area AHC}}{AB^2}$.

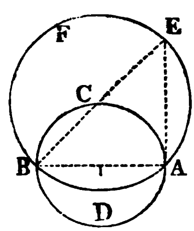
In like manner $\frac{\text{area AKE}}{AD^2} = \frac{\text{area ALH}}{AB^2}$, and $\frac{\text{area ADK}}{AD^2} = \frac{\text{area ABL}}{AB^2}$.

Adding these equals, we have,

$$\frac{\text{area } ADKEF}{AD^2} = \frac{\text{area } ABLHC}{AB^2} \therefore \frac{\text{area } ADKEF}{\text{area } ABLHC} = \frac{AD^2}{AB^2}.$$

—*See* TATE's *Geometry*, p. 92.

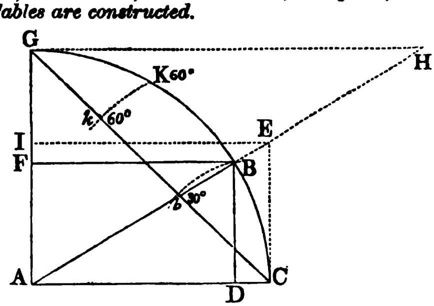
2°. *To make a circle the double or the half of another circle.*



Let AB be the diameter of ADBC; at A raise a perpendicular AE, and at B make the angle ABE equal to half a rt. angle; produce BC until it cuts AE: the square on BE will be double of the square on AB, and the circle of which BE is the diameter double of the circle of which AB is the diameter.

Again, let BE be the diameter of a circle; at E and B make angles each equal to half a rt. angle; and the square on AB will be one half of the square on BE; and the circle of which AB is the diameter one-half of the circle of which B E is the diameter.

2. By this 47th Proposition, the *Chords, Natural Sines, Tangents,* and *Secants of Trigonometrical Tables are constructed.*



With AC as radius desc. an arc CG, and from A raise a perpendicular AG; the arc CBKG being the measure of a rt. angle, is equal to 90°: join C and G, CG is the line of chords;—on which the chords of CB 30°, CK 60°, being inflected, Cb is the chord of 30°, Ck the chord of 60°. The sine of arc CB is BD, the tangent CE, the secant AE. The co-sine is FB, the co-tangent GH, the co-secant AH.

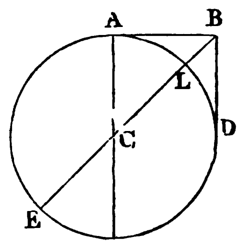
Let it be supposed that the Radius AB is divided into 1,000,000 parts, and that the arc BC is 30°. Since the Chord Ck of 60° is equal to the Radius AC; BD the sine of 30° shall be equal to the half of AC, or 500000, in the rt. angled triangle ADB. Now $AB^2 = AD^2 + BD^2$; and $AB^2 - BD^2 = AD^2$ or BF^2, the Sine of the complement: substituting the numbers we have $\sqrt{(1000000^2 - 500000^2)} = 866025 = FB$. Next, as the triangles ABD, AEC are equi-angular, we have the proportion AD : BD :: AC : CE; therefore, the tangent of 30° $CE = \frac{BD \cdot AC}{AD}$, or, $\frac{500000 \times 1000000}{866025} = \frac{50000000000}{866025} = 577350$. Then $AC^2 + CE^2 = AE^2$, and AE is the secant of 30°;—therefore $\sqrt{(1000000^2 + 577350^2)} = 1154703$ the Natural Secant for an arc of 30°.

3. *Arithmetically,* when the given numbers are 3, 4, 5, or their equimultiples, *the sum of the squares of the two less is equal to the square of the greater,* as $(6 \times 6) + (8 \times 8) = 10 \times 10 = 100$, and when any two are given

we can find the third exactly: but with respect to all other numbers, though the sum of the squares of any two numbers always equals or constitutes the square of a number greater than either, we cannot attain that number with perfect accuracy; excepting in the case of *right triangular numbers*, all we can do is to approach its value by increasing the number of decimal places in the root. Thus $(5 \times 5) + (8 \times 8) = 25 + 64 = 89$; but the square root of 89 is 9·433981, &c.—*See the Introduction*, § vi., *on Incommensurable Quantities.*

Right triangular numbers may be found thus:—

Put n any odd number, then $\frac{n^2-1}{2}$ = the second number, and $\frac{n^2+1}{2}$ = the third number: thus, take 7, then $\frac{49-1}{2} = 24$, and $\frac{49+1}{2} = 25$. Now $(24 \times 24) + (7 \times 7) = 576 + 49 = 25 \times 25 = 625$; the rt. triangular numbers being 7, 24, and 25.



4. The height of any elevation on the earth's surface is so small when compared with the earth's diameter, that for practical purposes, as levelling, and ascertaining the height of mountains, we may consider the earth's actual diameter, and the diameter + the elevation, as the same quantity, *i. e.*, BE and LE not sensibly to differ; nor the arc AL from the horizontal level AB. We assume LE to be 7960 miles, or that we may have an easier number 8000 miles. If we take AB one mile, then $BL = \frac{1}{8000}$ part of a mile, or nearly 8 inches; *i. e.*, for every mile of survey, the surface or curvature of the earth is 8 inches below the horizontal level.

5. *Heights and Distances from the curvature of the earth* are computed by Prop. 47, from the principle established in Prop. 16, bk. iii., that the tangent AB is perpendicular to the radius CA of the arc AL. Then if AB be required, we have $\sqrt{(LC + LB)^2 - AC^2} = AB$: if LB, the formula is $\sqrt{AB^2 + AC^2} = BC$, and $BC - LC$ or $AC = BL$.

Example 1. *Given* BL, *the height of the Peak of Teneriffe; what will be the radius of its horizon, or the distance at which it may be seen?*

Here $CB = CL + LB$. And $\sqrt{CB^2 - AC^2} = AB$ the horizontal radius.
Or, $4002^2 - 4000^2 = 16016004 - 16000000 = 16004$. And $\sqrt{16004} = 126$ miles $= AB$.

Ex. 2. *A meteor* B *is seen over a distance from* A *to* D *of* 200 *miles: required its height.*

Here $BL = BC - LC$ or AC. And $\sqrt{AC^2 + AB^2} = CB$.
Or, $\sqrt{4000^2 + 100^2} = \sqrt{16010000} = 4001{\cdot}24$ miles. And $4001{\cdot}24 - 4000 = 1{\cdot}24$ miles height of the meteor.

Ex. 3. *A fountain* B *one mile from* A, *is observed from* A *to have the same apparent level: how much is* B *above* A? i. e., *how much is* B *further from the earth's centre than* L *or* A?

Here $BC - LC = BL$. And $\sqrt{(4000^2 + 1^2)} = 4000.0001255 = BC$: then $4000.001255 - 4000 = .0001255$ of a mile = 8 inches nearly.

By Prop. 36, bk. iii., the square of the tangent AB equals the rectangle of BL into BE; and as in levelling the distances are usually small, $AB^2 = BL \times EL$ nearly.

When AB is 1 mile, BL is $\frac{2}{3}$ of 1 foot, or 8 inches;
" AB is 2 miles, BL is $\frac{2}{3}$ of 4 feet, or 32 "
" AB is 3 miles, BL is $\frac{2}{3}$ of 9 feet, or 6 feet;
" AB is 4 miles, BL is $\frac{2}{3}$ of 16 feet, or 10·6 feet.

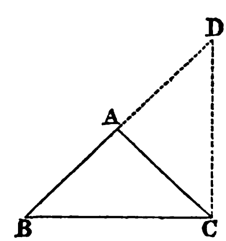
Thus two-thirds of the square of the number of miles that the level is long, gives the height of B above A in feet, or what the horizontal level differs from the level of the earth's curvature.

Prop. 48.—Theor.

If the squares described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by the two sides is a rt. angle.

Construction.—P. 11. To draw a st. line at rt. angles to a given st. line from a given point in it.
P. 3. From the greater to cut off a part equal to the less.
Pst. 1. A line may be drawn from one point to another.

Demonstration.—Ax. 2. If equals be added to equals, the wholes are equal.
P. 47. In a rt. angled triangle the square on the side subtending the rt. angle is equal to the sum of the squares of the sides containing the rt. angle.
Ax. 1. Magnitudes equal to the same magnitude are equal to each other.
P. 8. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them of the other.



Exp.	1	Hyp.	In triang. ABC let the square on BC = the sum of the squares on AB and AC;
	2	Concl.	then the ∠ BAC is a rt. angle.
Cons.	1	P. 11.	At A draw AD at rt. angles with AC;
	2	P.3 & Pst.1	make AD = AB, and join DC.

DEM.	1	C. 2.	∵ DA = AB, ∴ DA square = AB square:
	2	Add.	Let the square on AC be added to each;
	3	Ax. 2.	then the squares on DA and AC = the squares on AB and AC:
	4	C. 1.	But DAC is a rt. angle;
	5	P. 47.	∴ the square on DC = the squares on AD and AC.
	6	Hyp.	Also the square on BC = the squares on AB and AC;
	7	Ax. 1.	∴ the square on DC = BC square, and DC = BC:
	8	C. 2 & D. 7.	Thus in △s DAC, BAC, AD = AB, DC = BC, and AC is common;
	9	P. 8.	∴ ∠DAC = ∠BAC:
	10	C. 1.	But DAC is a right angle;
	11	Ax. 1.	∴ BAC is a right angle.
	12	Recap.	*Therefore, if the squares described*, &c.

Q.E.D.

SCHOLIUM.—The 48th is the converse of the 47th Proposition, and may be extended thus:—The vertical angle of a triangle is less than, equal to, or greater than, a rt. angle, as the square on the base is less than, equal to, or greater than, the sum of the squares of the sides.

REMARKS ON BOOK I.

1. IT will have been seen that the First Book is founded entirely on the Definitions, Postulates, and Axioms;—the *first* fixing the meaning of the terms employed; the *second* assigning the instruments that may be used; and the *third* setting forth the principles on which the comparisons and arguments are conducted. In a few instances, for the illustration of certain propositions, other principles, not belonging to the first book, have been assumed;—but these are to be regarded in their proper light, *not* as strict proofs, *but* as methods of explanation.

2. A few only of the properties of the circle are mentioned: those of the straight line and rectilineal angle are subservient to the proof of the properties of the triangle; and all rectilineal figures are either triangles, or may be resolved into triangles. The *First Book* therefore may in general terms be described as treating of the *Geometry of Plane Triangles.*

3. Excluding the Definitions, Postulates, and Axioms, it is not unusual to make a three-fold division of the contents of this Book. The *first* part, extending from the 1st Prop. to the 26th, unfolds the properties of triangles; the *second*, from Prop. 27 to 32, those of parallel lines; and the *third*, from Prop. 33 to 48, those of parallelograms, of course including the square.

4. The most important Propositions are,—*three*, namely, Props. 4, 8, and 26, containing the *criteria*, or conditions of equality between triangles; *one*, Prop. 32, the equality of the exterior angle to the two interior and opposite angles, and of the three interior angles of every triangle to two right angles; *one*, Prop. 41, the proportion of the parallelogram to the triangle on the same base and between the same parallels; and *one*, Prop. 47, the relation between the hypotenuse and the sides about a right angle. These propositions at least must be thoroughly mastered, not by committing them to memory, but by becoming so perfectly familiar with the principles contained in them, and with the connexions which exist between the arguments or reasonings employed, as never to feel at a loss for the demonstration, however diversified may be the figures constructed, nor even though no figure at all be drawn. The great aim should be to understand, and as a means to this, to follow up each proposition regularly through all its gradations, and verify it by its appropriate proofs.

GRADATIONS IN EUCLID.

BOOK II.

CONTAINING THE PROPERTIES OF RIGHT-ANGLED PARALLELOGRAMS, OR RECTANGLES.

In this Book, the relations will be investigated between the rectangles formed by the segments of straight lines, or of lines produced. When a line is cut or divided at any point, the segments are the portions between the point and the extremities of the line; when that point is within the extremities, the line is cut *internally;* when the point assumed is without the given line, and the line has to be lengthened, it is cut *externally,*—the production of the line in this case containing the point of section. If a line is cut internally, the line is the *sum* of the segments; but if cut externally, the line is their *difference.*

The subject of Geometry being magnitude and not number, it is necessary, as we have said (p. 20), to discriminate between the Geometrical conception of a rectangle, and the Algebraical or Arithmetical representation of it: yet the latter, as illustrative of the Geometrical truth, will materially assist the former,—our ideas of number being more definite than our ideas of space. Accordingly, to each of the Propositions will be appended, what some have named, though loosely, the Algebraical or Arithmetical proof.

The numerical area of a rectangle is obtained by supposing the two sides containing the rectangle to be divided into a number of linear units of the same kind, as inches, feet, &c., and then multiplying the units in one side by the units in the other; the product represents the Area or enclosed space.

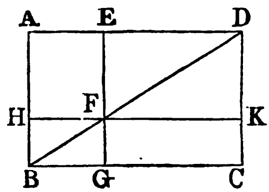
Of the two sides, one is considered as the *base,* the other as the *altitude;* and they may be represented by the letters b and a;—thus the formula for the area of a rectangle will be ab; and for that of a triangle $\frac{ab}{2}$ or $\frac{1}{2}\,ab$; and for a square a^2 or b^2, according to the side taken,—the sides in this case being equal.

DEFINITIONS.

1. Every right-angled parallelogram, or rectangle, is said to be contained by any two of the st. lines which contain one of the right angles.

The rectangle is contained by any two conterminous sides.

2. In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called the Gnomon. Thus the parallelogram HG, together with the complements AF, FC is the gnomon; which is more briefly expressed by the letters AGK or EHC, which are at the opposite angles of the parallelograms which make the gnomon.



AXIOM.

"The leading idea, which runs through the demonstrations of the first eight propositions, is the obvious axiom, founded on the 8th Axiom, bk. i., that the whole area of every figure, in each case, is equal to all the parts of it taken together."—POTTS' *Euclid*, p. 68.

N.B.—The Propositions, &c., required for the Construction and Demonstration will not in every instance be given. The learner is supposed to be familiar with most of them.

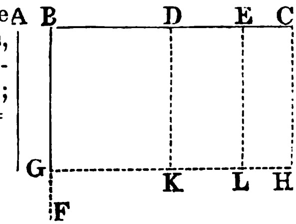
PROP. 1.—THEOREM.

If there be two st. lines, one of which is divided into any number of parts, the rectangle contained by the two st. lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.

CONS.—11. I. At a point in a st. line to draw a right angle.
3. I. From the greater of two lines to cut off a part equal to the less.
31. I. Through a point to draw a st. line parallel to a given st. line.

DEM.—34. I. The opposite sides and angles of parallelograms are equal.
Ax. 8. Magnitudes which coincide are equal: *i. e.*, the whole area of every figure, in each case, is equal to all the parts of it taken together.

Exp.	1	Hyp.	Let A and BC be the two lines, BC being divided in D & E;
	2	Concl.	then $\square$ A.BC = $\square$ A.BD + $\square$ A.DE + $\square$ A.EC.



Cons.	1	11. I.	At B draw BF at rt. angles to BC;
	2	3. I.	make BG = A:
	3	31. I.	through D, E, and C draw DK, EL, and CH $\parallel$s BG, and through G, GH $\parallel$ BC;
	4	Concl.	then $\square$ BH = $\square$s BK + DL + EH.
Dem.	1	Def. 1 & C.	$\because$ BH is contained by the lines GB, BC, of which GB = A;
	2	Concl.	$\therefore$ $\square$ BH = $\square$ A.BC:
	3	Def. 1 & C.	Also, $\because$ BK is contained by GB, BD, of which GB = A;
	4	Concl.	$\therefore$ $\square$ BK = $\square$ A.BD:
	5	C. & 34. I.	And $\because$ DL is contained by DK, DE, of which DK = GB = A;
	6	Concl.	$\therefore$ $\square$ DL = $\square$ A.DE:
	7	Sim.	In like manner $\square$ EH = $\square$ A.EC:
	8	Ax. 8.	$\therefore$ $\square$ A.BC = A.BD + A.DE + A.EC.
	9	Recap.	*Wherefore, if there be two st. lines, one of which,* &c. Q.E.D.

Cor. 2 A. $\frac{1}{2}$ BC; or 3 A. $\frac{1}{3}$ BC; or 4 A. $\frac{1}{4}$ BC, &c. = A.BC.

Sch.—The propositions of this Book may be verified by Algebra and by Arithmetic; and in doing this we shall first state the Hypothesis algebraically and numerically, and then separately give, what are denominated, the *Algebraic* and *Arithmetical* Proofs.

Alg. & Arith. Hyp.—Let $A = a = 6$; $BC = b = 10$; $BD + DE + EC = m + n + p = 5 + 3 + 2 = 10$.

Alg. $b = m + n + p$	*Arith.* $10 = 5 + 3 + 2$
$(\times a)$ $ab = am + an + ap$	$(\times 6)$ $6 \times 10 = (6 \times 5) + (6 \times 3) + (6 \times 2)$
	or, $60 = 30 + 18 + 12$

Use and App.—One of the methods of *Demonstrating the Rule for the Multiplication of numbers* depends on this proposition.

Let A represent 8, and BC 54. We cut or separate the number 54 into as many parts as there are digits: for example, 50 + 4; each part is multi-

plied by 8; the one part $4 \times 8 = 32$, and the other part $50 \times 8 = 400$. Now all the partial products make up the whole product; therefore $(4 \times 8) + (50 \times 8) = 54 \times 8$; or $32 + 400 = 432$.

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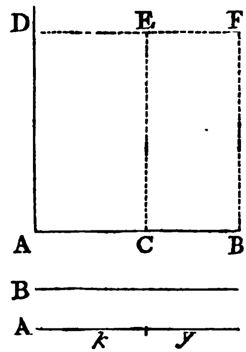
Prop. 2.—Theor.

If a st. line be divided into any two parts, the rectangles contained by the whole line and each of the parts, are together equal to the square of the whole line.

Cons.—46. I. On a given st. line to describe a square.
31. I. Through a given point to draw a parallel to a given st. line.

Dem.—Def. 30. I. Of four-sided figures a square is that which has all its sides equal, and all its angles rt. angles.
Ax. 1. Magnitudes which are equal, &c.



Exp.	1	Hyp.	Let AB be divided into any two parts in C;
	2	Concl.	then the □s AB.AC + AB.CB = the sq. on AB.
Cons.	1	46. I.	On AB describe the square ADFB;
	2	31. I.	and through C draw CE ‖ AD or BF;
	3	Concl.	then □s DC + EB = the square DB.
Dem.	1	Cons.	∵ AF = AE + CF, and AF = AB square;
	2	Def. 30. I.	also, ∵ AE is contained by AD, AC, of which AD = AB;
	3	Ax. 1.	∴ □ AE = AB.AC:
	4	Cons.	And ∵ CF is contained by BF, CB, of which BF = AB;
	5	Ax. 1.	∴ □ CF = AB.CB:
	6	D. 3 & 5.	Therefore □s AB.AC + AB.CB = the square on AB.
	7	Recap.	*If, therefore, a st. line be divided into any two parts, &c.* Q.E.D.

***Alg. & Arith. Hyp.*—Let $AB = a$ units $= 9$, and $AC + CB = m + n = 5 + 4$.**

Alg. Then $m+n=a$	*Arith.* Then $5+4=9$
$(\times a)$ $\therefore am+an=a\times a$, or a^2	$(\times 9)$ $\therefore 45+36=9\times 9=81$

SCH.—1. "There is no necessity for the absolute construction of the rectangles, to establish the relations they express."—LARDNER's *Euclid*, p. 66.

Thus, Given the line $A = {}^*k+y$, to prove that $A^2 = Ak+Ay$.
Take the line $B=A$; then $B.A = Bk+By = Ak+Ay$.

2. "The object of most of the propositions of this book is, to determine the relations between the rectangles under the parts of *divided lines.* We shall first confine our attention to a finite line divided into two parts."

"In this case there are three lines to be considered,—1st, the whole line, expressed by W; 2nd, the greater part, by P; 3rd, the less part, by p: then $W^2 = (W.P+W.p)$.

But the two parts may be considered as two independent lines, L, and l; then the whole line is their sum, S; and $S^2 = S.L+S.l$: and D being the difference, or $L—l$, $L^2 = L.l+L.D$.

USE AND APPLICATION.—*Numerical Multiplication* may also be proved by this process: for if a number be divided into its parts, the square of the number, which is the product of the number multiplied into itself, equals the sum of the products of each part into the undivided number. In the same way in *Algebraical equations*, in which a quantity may be represented by a, and its parts by $m+n$; if both sides of the equation are multiplied by the quantity a, then $a\times a$ or $a^2 = (m\times a)+(n\times a) = ma+na$.

* The wood engraver substituted k for x in the figures; hence the use of k.

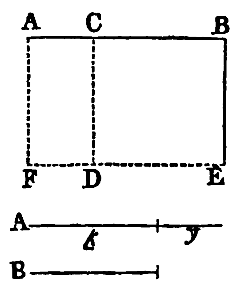
PROP. 3.—THEOR.

If a st. line be divided into any two parts, the rectangle contained by the whole line and one of the parts, is equal to the rectangle contained by the two parts together with the square of the aforesaid part.

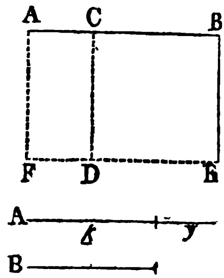
CONS.—46. I., Pst. 2, and 31. I.

DEM.—Def. 1. II., Def. 30. I., and Ax. 1.

EXP.	1	Hyp.	Let the st. line AB be cut in C;
	2	Concl.	Then □ AB.BC = □ AC.CB + the square on BC.
CONS.	1	46.I. Pst. 2	On BC draw a square CDEB, and produce ED to F;
	2	31. I.	through A draw AF ‖ CD or BE;



Cons.	3	Concl.	then □ AE, *i.e.*, AB.BC = □ AD + square DB.
Dem.	1	Def. 1. II. & 30. I.	∵ □ AE is contained by AB.BE, of which BE = BC;
	2	Ax. 1.	∴ □ AE = AB.BC:
	3	Def. 1. II. & 30. I.	Also, ∵ □ AD is contained by AC.CD, of which CD = BC;
	4	Ax. 1.	∴ □ AD = AC.BC:
	5	Hyp.	And □ DB = the square on CB:
	6	D. 4 & 5.	∴ □ AB.BC = □AC.BC + BC².
	7	Recap.	*Thus, if a st. line be divided into any two parts*, &c. Q.E.D.



Alg. & Arith. Hyp.—Let AB $= a = 9$; BC $= m = 6$; and AC $= n = 3$.

Alg. Then $a = m + n \quad (\times m)$
$\therefore ma = m^2 + mn$

Arith. Then $9 = 6 + 3 \quad (\times 6)$
$\therefore 54 = 36 + 18$

Or, Let A be a line divided into k and y, and B another line $= k$;
Then (1. II.) $A \cdot B = B \cdot k + B \cdot y$. But (*Hyp.*) $B = k$.
therefore $B \cdot k = k^2$; and $B \cdot y = k \cdot y$. Thus $A \cdot B = k^2 + k \cdot y$.

Cor. 1. $A^2 - B^2 = (A + B) \cdot (A - B)$; or $(k + y)^2 - k^2 = (k + y + k) \cdot (k + y - k) = (2k + y) \cdot y$; or $81 - 36 = 45 = 15 \times 3$.

2. $A^2 - B^2$ is greater than $(A - B)^2$ by twice $B \cdot (A - B)$; or $(k + y)^2 - k^2$ is greater than $(k + y - k)^2$ or y^2 by $2k \cdot (k + y - k)$ or $2k \cdot y$; or $81 - 36$ or 45 greater than 9 by twice 6×3, or 36.

Use and Application.—Multiplication of numbers may also be proved by this 3rd Prop.; for if a number, as 56, has to be multiplied by another, as 7, if the number, as 56, be separated into two parts, of which the multiplier 7 shall be one part, then on taking the square of the multiplier 7 one part, and multiplying the other part 49 by the same multiplier, the product will equal 7 times 56. Thus 56×7, or $392 = (7 \times 7) + (49 \times 7) = 49 + 343$.

Prop. 4.—Theor.

If a st. line be divided into any two parts, the square of the whole line equals the squares of the two parts, together with twice the rectangle contained by the parts.

Cons.—46. I., Pst. 1, and 31. I.

Dem.—29. I. If a line falls on two parallel lines, it makes the alternate angles equal, and the ext. angle equal to the int. opposite angle, and the two interior angles equal to two rt. angles.

Def. 30. I. A square has its sides equal, and its angles rt. angles.

5. I. The angles at the base of an isosceles triangle are equal, and if the equal sides be produced, the angles on the other side of the base shall be equal.

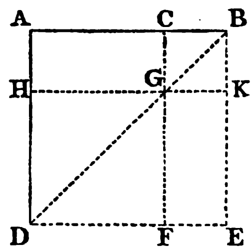
Ax. 1. Magnitudes, &c.

6. I. If two angles of a triangle be equal to one another, the sides also which subtend the equal angles shall be equal to one another.

Ax. 3. If equals be taken, &c.

34. I. The opposite sides and angles of parallelograms are equal to one another, and the diagonal bisects them.

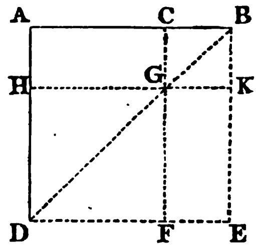
43. I. The complements of the parallelogram which are about the diameter of any parallelogram are equal to one another.



Exp.	1	Hyp.	Let the line AB be divided at C;
	2	Concl.	then the square on AB = the squares on AC and CB, together with twice the □AC.CB.
Cons.	1	46. I. Pst. 1	On AB construct the square ADEB, and join DB;
	2	31. I.	through C draw CGF ∥ AD or BE, and through G, HK ∥ AB or DE;
	3	Concl.	then AE = HF + CK + 2 AG.
Dem.	1	C.2 & 29.I.	∵ BD falls on the ∥s CF, AD, ∴ the ext. ∠ BGC = the int. ∠ ADB;
	2	C.1, D.30.I	But ADEB being a square, AB = AD:
	3	5. I. Ax. 1.	And ∵ ∠ ADB = ∠ ABD, and ∠ BGC = ∠ CBG;
	4	6. I.	∴ BC = CG:
	5	34. I. D. 4.	But GK = BC, CG = BK, and ∴ BC = GC = GK = BK;
	6	Ax. 1.	and ∴ the fig. CGKB is equilateral.
	7	C. 2, 29. I.	Again, ∵ CB meets the ∥s CG, BK, the ∠s KBC, GCB = two rt. angles:
	8	D.30, Ax.3	but KBC is a rt. angle, ∴ GCB is a rt. angle;
	9	34. I.	and ∴ the ∠s opposite, CGB, GKB, are rt. angles;

Dem.

10	Concl.	∴ also CGKB is rectangular:
11	D. 6 & 10.	∴ CGKB is a square and on CB.
12	Sim. 34. I.	For the same reason HF is a square on HG, which = AC;
13	D. 11 & 12	∴ HF and CK are squares on AC and CB.
14	43. I.	And ∵ the compl. AG = the compl. GE;
15	D. 4.	and ∵ GC = CB, and □ AG = AC·GC, or AC·CB;
16	Ax. 1.	∴ □ GE = □ AC·CB;
17	D. 15 & 16	and ∴ □s AG and GE together equal twice AC·CB:
18	D. 13.	And HF, CK are squares on AC and CB;
19	D. 17 & 18	∴ HF + CK + AG + GE = $AC^2 + BC^2 + 2\ AC \cdot CB$.
20	Const.	But HF, CK, AG, and GE make up ADEB the square on AB;
21	Ax. 1.	∴ $AB^2 = AC^2 + BC^2 + 2\ AC \cdot CB$.
22	Recap.	*Wherefore, if a st. line be divided into any two parts, &c.* Q.E.D.



Alg. & Arith. Hyp.—Let AB = a = 12; AC = m = 8; and BC = n = 4.

Alg. Then $a = m + n$
Squaring, $a^2 = (m+n)^2$, or $m^2 + 2mn + n^2$

Arith. Then $12 = 8 + 4$
Squaring, $12^2 = (8+4)^2 = 144 = 64 + 64 + 16$

Or, Let the line be made up of the parts $k + y$,

Then (2. II.) $A^2 = A \cdot k + A \cdot y$. But (3. II.) $A \cdot k = k^2 + k \cdot y$: And $A \cdot y = y^2 + k \cdot y$.

Hence $A^2 = k^2 + y^2 + 2k \cdot y$.

Cor. 1.—*The parallelograms about the diameter of a square are also squares.*

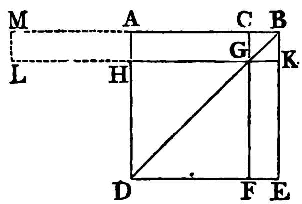
2.—*The square of a line is four times the square of its half;* for AC being equal to CB, $AC^2 + CB^2 = 2\ AC^2$ or $2\ CB^2$; and the rectangle AC·CB is the same as AC^2 or CB^2. Thus $AB^2 = 4\left(\frac{AB}{2}\right)^2$, *i. e.*, in the numerical example $12 \times 12 = 4\,(6 \times 6) = 144$.

3.—*Half the square of a line is equal to double the square of half the line;* thus $(\frac{AB}{2})^2 = 2\,(\frac{AB}{2})^2$ or $\frac{12 \times 12}{2} = 2\ (6 \times 6)$; or $\frac{144}{2} = 2 \times 36 = 72$.

4.—This proposition may also be employed for a line divided into any number of parts; then, *the square of the line will be equal to the sum of the squares of all the parts, together with double the rectangle under every distinct pair of parts.* Let $AB = k + y + z$; then $AB^2 = k^2 + y^2 + z^2 + 2\ (k \cdot y + y \cdot z + k \cdot z)$. Take a number $12 = 6 + 4 + 2$; then $12 \times 12 = (6 \times 6) + (4 \times 4) + (2 \times 2) + 2\ \{(6 \times 4) + (4 \times 2) + (6 \times 2)\}$; or $144 = 36 + 16 + 4 + 2\ (24 + 8 + 12)$.

USE AND APP.—1. In Algebra, the *square of a binomial*, that is, of a quantity made up of two terms, as $k + y$, is identical with the squares of the parts added to twice the product of the parts; thus $(k+y)^2 = k^2 + y^2 + 2\,k \cdot y$.

2. This Proposition points out a *practical way of extracting the Square root of a number.*



Let a number 144 be represented by the square AE, and its square root by the line AB. The square root of every number containing an even number of digits consists of half that even number of digits; and the square root of every number containing an odd number of digits, consists of $\frac{n+1}{2}$ digits. We know therefore that AB the line required will be represented by a number consisting of two digits. Suppose AB divided at C, so that AC represents the first digit, and CB the second. Seek the root of the first figure in 144, namely, of 100,—and it is found to be 10; the square of 10 therefore is represented by HF; HF being removed or subtracted, there remains 44 for the sum of the rectangles, AG, FK and of the square CK. But as the gnomon AKF is not convenient for use, the rectangle KF may be removed and placed as a continuation of AG, as MLHA; the whole rectangle LB therefore contains the remainder 44. As AC equals 10, twice AC, or MC=20. We must therefore divide 44 by 20;—that is, to find the divisor we double the Root already found, namely 10, and say,—how many twenties are there in 44? We find there are two for the side BK; but since 20 was not the whole side MB, but only a part MC, the new figure in the quotient, 2, must be added to the divisor 20, making 20 + 2, or 22; and 22 is contained exactly in 44. The first digit in the root is 1; the second, 2; or together 12. Thus the Square of 144 is equal to the square of 10, to the square of 2, and to twice 20, which are the two rectangles comprehended under 2 and 10. It is for these reasons that the formula or rule given for extracting the square root of a number, requires the number to be separated into periods of two; the nearest square, and its root, to the left hand period are found,—and as twice the rectangle of the parts added to the square of the part to be found, make up the remainder,—for the division of that remainder we take twice the root, or rectangle already found, and to complete the divisor add to *that* twice the root the new term of the root: and so on, until the operation is completed.

PROP. 5.—THEOR.

If a st. line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts together with the square of the line between the points of section, is equal to the square of half the line.

CON.—46. I., Pst. 1, and 31. I.

DEM.—43. I. The complements of the parallelograms about the diameter of any parallelogram are equal to one another.

Ax. 2. If equals be added, &c.

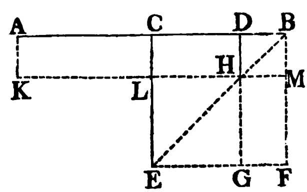
36. I. Parallelograms upon equal bases and between the same parallels are equal to one another.

Ax. 1. Things equal to the same, &c.

Def. 30. I. A square has its sides equal, and its angles rt. angles.

Cor. 4. II. The parallelograms about the diameter of a square are also squares.

34. I. The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them.



EXP.	1	Hyp. 1.	Let AB be divided into two equal parts in C,
		" 2.	and into two unequal parts in D;
	2	Concl.	then the ▭ AD.DB + the square on CD = the square on CB.
CONS.	1	46. I. Pst. 1.	On CB construct a square CEFB and join BE;
	2	31. I.	through D draw DHG ∥ CE or BF;
	3	" "	through H " KLM ∥ CB or EF;
	4	" "	& through A " AK ∥ CL or BM;
	5	Concl.	then ▭ AH + square on CD = square EB.

DEM.			
	1	43. I.	The complem. $CH =$ the complem. HF;
	2	Add. & Ax. 2.	adding DM to each, then $CM = DF$:
	3	Hyp.	But $\because AC = CB$;
	4	36. I. & Ax. 1.	$\therefore \square AL = \square CM$, and $AL = DF$;
	5	Add. & Ax. 2.	adding to each $\square CH$, then $\square AH = \square$s DF and CH.
	6	Def. 30. I. 4. II.	But $\because DH = DB$, and $\square AH$ is contained by $AD.DH$;
	7	Concl.	$\therefore \square AH$ also $= AD.DB$:
	8	Def. 2. II.	Also $\square$s DF and CH make the gnomon CMG;
	9	Ax. 1.	$\therefore CMG =$ the $\square AD.DB$:
	10	Add. Cor. 4. II	Add the square LG, that is the square on CD;
	11	Ax. 2.	then $CMG + LG = \square AD.DB + CD^2$:
	12	Cons.	But $CMG + LG$ make up $CEFB$, or the square on CB;
	13	Ax. 1.	$\therefore AD.DB + CD^2 = CB^2$.
	14	Recap.	*Wherefore, if a st. line be divided into two equal parts,* &c. Q.E.D.

Arith. & Alg. Hyp.—Let $AC = CB = a = 10$, and $AB = AC + CB = 2a = 20$, or $AB = AD + DB$, and let $CD = m = 6$. Then $AD = a + m = 10 + 6$, and $DB = a - m = 10 - 6$.

Alg. Now $m = \dfrac{(a+m)-(a-m)}{2}$

and $(a+m)(a-m) = a^2 - m^2$

$(+m^2) \therefore (a+m)(a-m) + m^2 = a^2$

Arith. $6 = \dfrac{(10+6)-(10-6)}{2} = \dfrac{12}{2}$

& $(10+6)(10-6) = 100 - 36 = 64$

$(+36) \therefore (16 \times 4) + 36 = 100$

COR.—It is manifest that *the difference of the squares of two unequal lines,* AC, CD, *is equal to the rectangle contained by their sum and difference*;—*i. e.*, $AC^2 - CD^2 = (AC + CD)(AC - CD)$.

From the square EB take the square DM, there will remain EM and CH; if HF be taken from EM and placed by the side of CH, as AL, then CH and HF become AH, and AH is contained by $AC + CD$ or AD, and by $AC - CD$ or DB: hence $AC^2 - CD^2 = (AC + CD)(AC - CD)$.

Since $AC=a=10$ is half the sum of $AD+DB$, and $CD=m=6$ half their difference, the corollary may be thus expressed:—"The rectangle contained by two st. lines AD and DB, is equal to the difference of the squares of half their sum AC, and half their difference CD;" *i. e.*, $AD.DB=AC^2-CD^2$, or $(a+m)\ (a-m)=a^2-m^2$.

Lardner's Corollaries to this Proposition are—1st. As the intermediate part CD diminishes, the rectangle increases, and *vice versâ*. The rectangle is a *maximum* when AB is bisected by D, its value being $\frac{(AB)^2}{2}$.—2nd. The greater the rectangle is, the less will be the sum of the squares of the parts; the sum of the squares of the parts being at a *minimum* when the line CB is bisected in D.—3rd. Of all rectangles having the same perimeter, the square contains the greatest area.—4th. Of all rectangles equal in area, the square is contained by the least perimeter.—5th. If a perpendicular be drawn from the vertex of a triangle to the base, the rectangle under the sum and difference of the sides is equal to the rectangle under the sum and difference of the segments.—6. The difference between the squares of the sides of a triangle, is equal to twice the rectangle under the base and the distance of the perpendicular from the middle point.—These Corollaries may be taken as the subjects of Geometrical Exercises.

SCH.—When a line is divided in two points *equally* and *unequally*, several linear magnitudes have to be considered;—1st. the whole line AB; 2nd, the equal segments AC and CB; 3rd, the unequal segments AD and DB; 4th, the intermediate part, that is, the part CD between the points of section.

The following are the principal properties connected with the equal and unequal division of a line:—1°. The intermediate part equals half the difference of the unequal parts; thus $CD=\frac{1}{2}(AD-DB)$, or $6=\frac{1}{2}(16-4)$. 2°. The greater segment equals half the sum added to half the difference; thus $AD=\frac{1}{2}(AD+DB)+\frac{1}{2}(AD-DB)$; or $16=\frac{1}{2}(20+12)$. 3°. The less segment equals half the sum *minus* half the difference; thus $DB=\frac{1}{2}(AD+DB)-\frac{1}{2}(AD-DB)$, or $4=\frac{1}{2}(20-12)$. 4°. The sum of two unequal lines equals twice the less added to the difference of the lines; thus $AD+DB=2DB+(AD-DB)$; or $16+4=(2\times 4)+(16-4)$.

USE AND APP.—1. By the Cor. to this Proposition *the difference between the squares* of two unequal numbers may be found *without squaring them*;—for the product of their sum and difference will equal the difference of the squares. Thus, if in a rt. angled triangle the hypotenuse is 100 feet, and the base 80, we may find the perpendicular without squaring the numbers;—for $(100+80)\times(100-80)=3600$, and $\sqrt{3600}=60=$the perpendicular.

2. We have also the means of finding *Quantities in Arithmetical Progression.* To be in Arithmetical Progression, quantities must increase or

diminish by a common difference: if there are three quantities, for instance, and the first exceeds the second by as much as the second exceeds the third, such quantities are in arithmetical progression,—the first and third being named the extremes, and the second the mean. Take, in the line A B, A D the greater segment = 16, D B the less = 4, and A C the half sum = 10. A C is the mean between A D and D B, the common difference being the intermediate part C D. In numbers we may take 16, 10, 4; these are in arithmetical progression, because their common difference is 6. Generally we affirm—"The arithmetical mean is half the sum of the extremes, and the common difference is half the difference of the extremes."

"The fifth proposition may then be announced thus:—The square of the arithmetical mean is equal to the rectangle under the extremes together with the square of the common difference."—LARDNER'S *Euclid*, p. 71.

3. This proposition is employed to establish one of the most important properties of lines cutting one another within a circle; as Prop. 35, bk. iii., "if two straight lines cut one another within a circle, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other."

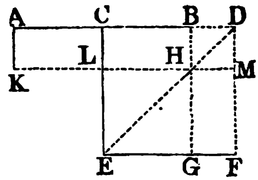
4. By this proposition the method may be demonstrated of *finding the value of an Adfected Quadratic Equation in Algebra*, that is, of an equation which contains the square and the first power of an unknown quantity. In such equations the square is not completed, being defective by the square of half the co-efficient of the second term. The problem to be solved is—Given, in the square C F, L G and 2 C H, required D M; *i. e.*, required the magnitude which will complete the square. The magnitude D M is the square on D B, and D B is equal to D H; and 2 D H·C D represents 2 C H. In the magnitude 2 D H·C D, 2 D H is the factor or co-efficient of C D; and if we take half of that co-efficient we have D H,—the square of which, D M, is the magnitude we are seeking to make up the square. To solve an adfected quadratic equation, as $x^2+4x=12$, we complete the square by adding to each side the square of half the co-efficient 4; then $x^2+4x+4=12+4=16$; extracting the square root, $x+2=4$ $\therefore x=2$.

PROP. 6.—THEOR.

If a st. line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square of half the line bisected, is equal to the square of the st. line which is made up of the half and the part produced.

CONS.—46. I., Pst. 1, and 31. I.

DEM.—43. I., Ax. 2, 36. I., Ax. 1, Def. 30. I., Cor. 4. II.



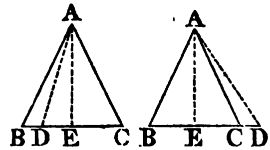
EXP.	1	Hyp.	Let AB be bisected in C & produced to D,
	2	Concl.	then □ AD·DB + the square on CB = the square on CD.
CONS.	1	46. I. Pst. 1.	On CD construct the square CEFD, and join DE;
	2	31. I.	through B draw BHG ‖ CE, or DF;
	3	" "	" H " KLM ‖ AD, or EF;
	4	" "	& " A " AK ‖ CL, or DM;
	5	Concl.	then □ AM + LG = square CF.
DEM.	1	H. & 36. I.	∵ AC = CB, □ AL = □ CH;
	2	43. I. & Ax. 1.	but □ CH = □ HF; ∴ □ AL = □ HF;
	3	Add. & Ax. 2.	Adding to each □ CM, then □ AM = the gnomon CMG.
	4	Cons., Cor. 4. II	But DM = DB; ∴ □ AM contained by AD·DM = □ AD·DB;
	5	Ax. 1.	and ∴ gnomon CMG = □ AD·DB:
	6	Add. Cor. 4. II	Adding LG the square on CB to each,
	7	Ax. 1 & 2.	then □ AD·DB and square on CB = CMG and LG:
	8	Const.	But CMG and LG make up CEFD the square on CD;
	9	Ax. 2.	∴ □ AD·DB + $CB^2 = CD^2$.
	10	Recap.	*Wherefore, if a st. line be bisected and produced, &c.* Q.E.D.

Alg. & Arith. Hyp.—Let AC or CB$=a=8$; AB$=2a=16$ & BD$=m=4$
AD$=2a+m=16+4$; and CD$=a+m=8+4$.

Alg. Now AD$=2a+m$ $(\times m)$
and therefore $(2a+m)\,m=2am+m^2$. (Add a^2)
therefore $(2a+m)\,m+a^2=a^2+2am+m^2$
But $a^2+2am+m^2=(a+m)^2$
therefore $(2a+m)\,m+a^2=(a+m)^2$

Arith. Now AD$=16+4$ $(\times 4)$
then $(16+4)\times 4=(2\times 8\times 4)+(4\times 4)=64+16$. (Add 8×8)
and $4\,(16+4)+64=64+64+16$
But $64+64+16=(8+4)^2=144$
therefore $4\,(16+4)+64=(8+4)^2=144$

Cor.—*If a line* AD *be drawn from the vertex* A *of an isosceles triangle to the base or its production, the difference between the squares of this line and the side of the triangle is the rectangle under the segments* BD × DC *of the base.*

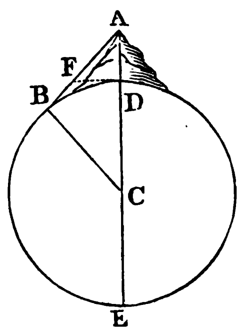


For (by Cor. 4, P. 47. I.) $AD^2 \sim AC^2 = CE^2 \sim DE^2$; but (by 5 and 6. II.) $CE^2 \sim DE^2 = BD \cdot DC$, $\therefore AD^2 \sim AC^2 = BD \cdot DC$.

N.B. If AD coincide with the perpendicular AE, the part DE will vanish.

Sch.—The algebraical results of Prop. 5 and 6 differ only in appearance, "arising from the two ways in which the difference between two unequal lines may be represented geometrically when they are in the same direction." In Prop. 5, the difference $DB = a - m = 10 - 6$, of the two lines $AC = a = 10$ and $CD = m = 6$, is exhibited by producing the less CD, and making $CB = CA$. Then $DB = AC$ or $CB - CD = a - m = 10 - 6 = 4$. In Prop. 6, the difference DB of the two unequal lines CD and CA, is exhibited by cutting off from CD the greater, a part CB equal to CA the less; then $DB = CD - CB$ or $CA = (a+m) - a = (8+4) - 8 = 12 - 8 = 4$.

Use and App.—Maurolico, a mathematician, and abbot of Messina, who died A.D. 1575, measured the diameter of the earth by making use of this Proposition.



From A the top of a mountain, the height of which AD is known, the angle CAB is measured, formed by EA and AB, a tangent to the circle in B, the limit of vision. In the rt. angled triangle ADF, the angles being obtained, the sides AF, FD will be found by Trigonometry; and FD equals FB: thus $AB = AF + FB$, and its square $= AB^2$.

Now (by Prop. 6. II.) ED being divided in C and produced to A, the rectangle $EA \cdot AD + CD^2 = AC^2$; and (by Prop. 18. III.) ABC is a right angle: therefore $AC^2 = AB^2 + BC^2$ or CD^2, and thus the rectangle $EA \cdot AD + CD^2 = AB^2 + CD^2$; take away the common part CD^2, and the rectangle $EA \cdot AD = AB^2$. Divide both sides of the equation by AD, and $EA = \frac{AB^2}{AD}$. Then $EA - AD = ED$ the diameter of the earth.

Example. *A mountain is* $2\frac{1}{2}$ *miles*, AD, *above the level*, FD, *of the sea; and the limit of vision, or* AB, *is* 141 *miles: required the earth's diameter* DE.

By the result just obtained $EA = \frac{AB^2}{AD}$, and $ED = EA - AD$.

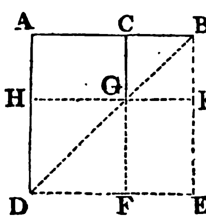
Therefore $\frac{AB^2}{AD}$ or $\frac{141 \times 141}{2 \cdot 5} = \frac{19881}{2 \cdot 5} = 7952 \cdot 4$ miles; and $7952 \cdot 4 - 2 \cdot 5 = 7949 \cdot 9$ the earth's diameter.

PROP. 7.—THEOR.

If a st. line be divided into any two parts, the squares of the whole line and of one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

CONS.—46. I., Pst. 1 and 31. I.

DEM.—43. I., Axs. 2 and 6, Def. 30. I., Cor. 4. II., Ax. 8, and 34. I.



EXP.	1	Hyp.	Let AB be divided into any two parts in C;
	2	Concl.	then the squares on AB and CB = twice $AB \cdot BC + AC^2$.
CONS.	1	46. I. Pst. 1.	On AB make a square ADEB, and join DB;
	2	31. I.	through C draw CF ∥ BE,
	3	" "	& " G " HK ∥ AB;
	4	Concl.	then the squares AE + CK = twice AK + HF.
DEM.	1	43. I.	∵ the compl. AG = the compl. GE;
	2	Add. & Ax. 1	on adding CK to each, AK = CE;
	3	Ax. 6.	∴ AK + CE = twice AK:
	4	Const.	But AK + CE make up AKF + CK;
	5	Ax. 1.	∴ AKF and CK together = twice AK:
	6	D. 30 I., 4. II	but BK = BC,
	7	Ax. 6.	twice AK = twice $AB \cdot BK$, and twice $AB \cdot BK$ = twice $AB \cdot BC$;
	8	Ax. 1 & 8.	∴ the gnom. AKF + CK = twice the rect. $AB \cdot BC$:
	9	Add. & 34. I	Adding to both the equals HF, *i. e.*, the square on HG or AC,
	10	Ax. 2.	then AKF + CK + HF = twice $AB \cdot BC$ and AC squared:
	11	Const.	But AKF + CK + HF make up the figures ADEB and CK,
	12	C. 1, 3.	and ADEB and CK are the squares on AB and CB;
	13	Concl.	∴ $AB^2 + BC^2$ = twice $AB \cdot BC + AC^2$.
	14	Recap.	*Wherefore, if a st. line be divided into any two parts*, &c. Q.E.D.

Alg. & Arith. Hyp.—Let $AB = a$ linear units $=16$; $AC = m = 9$ and $CB = n = 7$.

Alg. Then $a = m+n$; Squaring, $a^2 = m^2 + 2mn + n^2$.
(Add n^2) $a^2 + n^2 = m^2 + 2mn + 2n^2$.
But $2mn + 2n^2 = 2(m+n)n = 2an$,
therefore $a^2 + n^2 = 2an + m^2$.

Arith. $16 = 9 + 7$; Squaring, $256 = 81 + 126 + 49$.
(Add 49) $256 + 49 = 81 + 126 + 98$.
But $126 + 98 = 2(9+7)7 = 2(16 \times 7) = 224$,
therefore $256 + 49 = 222 + 81 = 305$.

Another form of stating the same result is,

Let $AB = a = 16$; $AC = b = 9$; and $BC = a - b = 16 - 9 = 7$.

$$\begin{array}{lcr} \text{then } AB^2 = a^2 & & \text{And } 2AB \cdot BC = 2a^2 - 2ab. \\ \underline{BC^2 = a^2 - 2ab + b^2} & & \underline{AC^2 = \qquad\qquad b^2} \\ \text{Sum } 2a^2 - 2ab + b^2 & = & 2a^2 - 2ab + b^2 \end{array}$$

Cor. 1.—If AB and BC be considered as two independent lines, AC being their difference, "*the sum of the squares of any two lines is equal to twice the rectangle under them together with the square of the difference;*" *i. e.*, $AB^2 + BC^2 = 2AB \cdot BC + AC^2$; or $100 + 64 = (2 \times 80) + 4$.

Cor. 2.—Hence and (by 4. II.) *the square of the sum of two lines, the sum of their squares, and the square of their difference, are in arithmetical progression,—the common difference being twice the rectangle under the sum.*

By 4. II. $(AB+BC)^2 = AB^2 + BC^2 + 2AB \cdot BC$; and by Cor. 7. II. $AB^2 + BC^2 = (AB - BC)^2 + 2AB \cdot BC$; or $AC^2 + 2AB \cdot BC$. Thus the common difference is $2AB \cdot BC$; therefore the quantities $(AB+BC)^2$, $(AB^2 + BC^2)$, and AC^2 are in arithmetical progression; as 324, 164, and 4,—the com. dif. being 160.

Prop. 8.—Theor.

If a st. line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts together with the square of the other part, is equal to the square of the st. line which is made up of the whole and that part.

CONS.—Pst. 2. A terminated st. line may be produced in a st. line.

3. I. From the greater line to cut off a part equal to the less.

46. I. On a given st. line to describe a square.

31. I. Through a given point to draw a st. line parallel to a given st. line.

DEM.—34. I. The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them.

Ax. 1. Magnitudes equal, &c.

36. I. Parallelograms upon equal bases and between the same parallels are equal.

43. I. The complements of the parellelogram which are about the diameter of any parallelogram, are equal to one another.

4. II. If a st. line be divided into any two parts, the square of the whole line equals the square of the two parts together with twice the rectangle contained by the parts.

Def. 30. I. A square is a four-sided figure having all its sides equal, and its angles rt. angles.

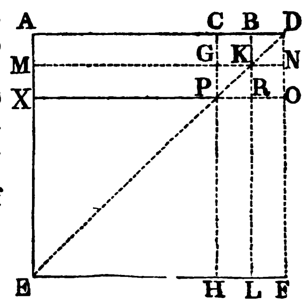
Cor. 4. II. The parallelograms about the diameter of a square are also squares.

Ax. 2. If equals be added, &c.

Ax. 8. Magnitudes which coincide are equal.

Ax. 6. Things double of the same are equal.

EXP.	1	Hyp.	Let AB be divided in the point C;
	2	Concl.	then four times AB·BC with the square on AC added = the square of (AB+BC).
CONS.	1	Pst. 2 & 3. I	Produce AB to D, so that BD = BC;
	2	46. I. Pst. 1	On AD describe the square AEFD, and join ED;
	3	31. I.	through B and C draw ||s BL, CH, to AE or DF, and cutting ED in the points K and P;
	4	" "	also through K and P, MGKN and XPRO ||s to AD or EF;
	5	Concl.	then AK + MR + (HR + BN) + NL + XH, fill up the figure AEFD.



DEM.	1	C. 1. & 34. I.	$\because$ CB = BD, CB = GK, and BD = KN;
	2	Ax. 1.	$\therefore$ GK = KN.
	3	Sim.	In like manner PR = RO:
	4	C. 1 & D. 2.	And $\because$ CB = BD, and GK = KN;
	5	36. I.	$\therefore$ □ CK = BN, and GR = □ RN:
	6	C. & 43. I.	But □s CK, RN are compls. of the rect. CO, and therefore CK = RN:
	7	D. 4.	also rect. BN = rect. GR;
	8	Ax. 1.	$\therefore$ □ BN = CK = GR = RN;
	9	Ax. 6.	and therefore the sum of these four = four times CK.
	10	C. 1, Def. 30 & 34. I.	Again, $\because$ CB = BD, BD = BK, and BK = CG;
	11	34 I. Cor. 4. II	and also $\because$ CB = GK, and GK = GP;
	12	Ax. 1.	$\therefore$ CG = GP.
	13	D. 12 & 2.	And $\because$ CG = GP, and PR = RO;
	14	C. 3 & 4.	and AX $\parallel$ CP, and PO $\parallel$ HF;
	15	36. I.	$\therefore$ rect. AG = MP. and PL = RF:
	16	Const.	But $\because$ MP, PL are the compls. of the □ MKLE,
	17	43. I. & Ax. 1	$\therefore$ MP = PL, and also AG = RF;
	18	D. 15 & 17.	$\therefore$ AG, MP, PL, and RF are all equal to one another;
	19	Ax. 1.	and the sum of the four = four times any one, as AG.
	20	D. 9.	But (BN + CK + GR + RN) = four times CK:
	21	Add. & Ax. 2	$\therefore$ the eight rectangles in the gnomon AOH = four times AK.
	22	Const.	Now rect. AK is contained by AB·BK, and AB·BK = AB·BC;
	23	D. 21.	$\therefore$ four times AK = four times AB·BC:
	24	"	But four times AK = the gnomon AOH;
	25	Ax. 1.	$\therefore$ four times AB·BC = AOH:
	26	Ad., Cor. 4. II	Add to each XH, that is the square on AC;
	27	34. I. & Ax. 2	then four times AB·BC + AC^2 = AOH + XH:
	28	Ax. 8.	But AOH + XH make up AEFD the square on AD;
	29	Concl.	$\therefore 4\,AB\cdot BC + AC^2 = AD^2 = (AB+BC)^2$.
	30	Recap.	*Wherefore, if a st. line be divided, &c.*

Q.E.D

Alg. & Arith. Hyp.—Given $AB=a=16$; $AC=m=10$, and $CB=n=6$.

Alg. Then $m+n=a$; taking n from each, $m=a-n$;	*Arith.* $10+6=16$; (-6) and $10=16-6$;
Squaring, $m^2=a^2-2an+n^2$	Squaring, $100=256-192+36$,
$(+4an)$ then $4an+m^2=a^2+2an+n^2$;	$(+4\times 96)$ then $384+100=256+192+36$;
But $a^2+2an+n^2=(a+n)^2$,	But $256+192+36=(16+6)^2$,
therefore $4an+m^2=(a+n)^2$.	therefore $384+100=(22)^2=484$.

We obtain the same result in another form:

Let $AB=a=16$, $BC=b=6$, and $AC=a-b=16-6=10$;

then $4AB\cdot BC=4ab$; And $(AB+BC)^2=a^2+2ab+b^2$

$AC^2=a^2-2ab+b^2$

Sum $a^2+2ab+b^2$ = Sum $a^2+2ab+b^2$

SCH.—1. The Proposition may be otherwise expressed:—"*the square of the sum of two lines is equal to four times the rectangle under them together with the square of their difference;*" thus $(AB+CB)^2=4AB\cdot CB+(AB-CB)^2=(16+10)^2=4\ (16\times 10)+(16-10)^2$; or $676=640+36$.

2. Or, *four times the square of half the sum is equal to four times the rectangle under the lines together with four times the square of half the difference;*" thus, $4\ \frac{(AB+CB)^2}{2}=4\,AB\cdot CB+4\ \frac{(AB-CB)^2}{2}=4\times 169=(4\times 160)+(4\times 9)=676$.

USE AND APP.—The Principles established in Props. 6, 7, and 8 are applied to Algebra and to various operations connected with the extraction of the Square root.

—

PROP. 9.—THEOR.

If a st. line be divided into two equal parts, and also into two unequal parts, the squares of the two unequal parts are together double of the square of the half line and of the square of the line between the points of section.

CONS.—11. I. To draw a st. line at rt. angles to a given line at a given point in it.
3. I. From the greater of two lines to cut off a part equal to the less.
Pst. 1. A line may be drawn, &c.
31. I. Through a given point to draw a line parallel to a given line.

DEM.—5. I. The angles at the base of an isosc. triangle are equal.
32. I. If a side of any triangle be produced, the ext. angle equals the two int. and opp. angles; and the three int. angles of every triangle equal two rt. angles.

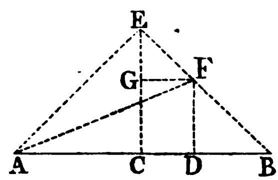
29. I. A line falling on two parallel lines makes the alt. angles equal, and the ext. angle equals the int. and opp. angles upon the same side; and the two int. angles upon the same side equal two right angles.

6. I. If two angles of a triangle be equal to one another, the sides opposite the equal angles shall be equal to one another.

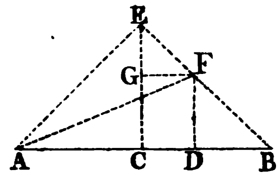
47. I. In every rt. angled triangle the square on the side subtending the rt. angle is equal to the sum of the squares on the sides containing the rt. angle.

34. I. The opposite sides and angles of parallelograms equal one another, and the diagonal bisects them.

Ax. 1. Magnitudes equal to the same are equal.



Exp.	1	Hyp. 1.	Let AB be divided into two equal parts at C,
		" 2.	and into two unequal parts at D;
	2	Concl.	then $AD^2 + DB^2 = 2\ (AC^2 + CD^2)$.
Cons.	1	11. I.	From C draw CE at rt. angles to AB:
	2	3. I. & Pst. 1	make CE = AC or CB, and join EA, EB;
	3	31. I.	through D draw DF ∥ CE, and meeting EB in F;
	4	31. I. & Pst. 1	and through F draw FG ∥ AB, and join AF;
	5	Concl.	then the squares on AD and DB = twice the squares on AC and CD.
Dem.	1	C. 2 & 5. I.	∵ AC = CE, ∴ ∠ EAC = ∠ AEC;
	2	C. 1 & 32. I.	and ACE being a rt. angle, ∴ ∠s AEC and EAC = a rt. angle;
	3	D. 1 & 32. I.	and ∠ AEC being = ∠ EAC, each is half a rt. angle.
	4	Sim.	So, ∵ ∠ CEB = ∠ EBC half a rt. angle, ∴ (AEC and BEC) or ∠ AEB is a rt. angle:
	5	D.4, 29.I.C.4	And ∵ GEF is half a rt. angle, and ∠ EGF = ECB a rt. angle;
	6	32. I.	∴ rem. ∠ EFG = half a rt. angle, and GEF = ∠ EFG;
	7	6. I.	and ∴ the side EG = the side GF.



DEM.	8	D. 4, 29. I.	Again, $\because$ FBD = half a rt. angle, and FDB = the int. $\angle$ ECB a rt. angle;
	9	32. I.	$\therefore$ rem. $\angle$ BFD is half a rt. angle, and $\angle$FBD = $\angle$BFD;
	10	6. I.	and $\therefore$ the side DF = the side DB:
	11	C. 2.	And $\because$ AC = CE, $\therefore AC^2 = CE^2$;
	12	Concl.	and $\therefore AC^2 + CE^2$ = twice AC^2.
	13	C. 1 & 47. I.	But ACE being a rt. angle, $AE^2 = AC^2 + CE^2$, or twice AC^2:
	14	D. 7.	Again, since EG = GF, the square on EG = that on GF; and $\therefore EG^2 + GF^2$ = twice GF^2:
	15	47. I.	but $EF^2 = EG^2$ and GF^2;
	16	34. I.	$\therefore EF^2$ = twice GF^2 which = twice CD^2;
	17	D. 13.	and from above AE^2 = twice AC^2;
	18	D. 13 & 15.	$\therefore AE^2 + EF^2$ = twice $(AC^2 + CD^2)$:
	19	D. 4 & 47. I.	But AEF being a rt. angle, $AF^2 = AE^2 + EF^2$;
	20	Ax. 1 & D. 18	and $\therefore AF^2$ = twice $(AC^2 + CD^2)$:
	21	C. & 47. I.	But ADF being a rt. angle, $AF^2 = AD^2 + DF^2$;
	22	Ax. 1.	and $\therefore AD^2 + DF^2$ = twice $(AC^2 + CD^2)$;
	23	D. 10.	and DF equalling DB, $AD^2 + DB^2$ = twice $(AC^2 + CD^2)$.
	24	Recap.	*Wherefore, if a st. line be divided into two, &c.* Q.E.D.

Alg. & Arith. Hyp.—Let AC, CB each $= a = 10$; AB $= 2a = 20$; and CD $= m = 4$.

Also, let AD $= a + m = 10 + 4$ and DB $= a - m = 10 - 4$.

Alg. Then $(a+m)^2 = a^2 + 2am + m^2$
and $(a-m)^2 = a^2 - 2am + m^2$
Adding $(a+m)^2 + (a-m)^2 = 2a^2 + 2m^2$

Arith. Then $(10+4)^2 = 100 + 80 + 16 = 196$
and $(10-4)^2 = 100 - 80 + 16 = 36$
Adding $(10+4)^2 + (10-4)^2 = 200 + 32 = 232$

Another form of the same Algebraical result is

$$\begin{array}{ll} AD^2 = a^2+2am+m^2 & \text{And } 2AC^2=2a^2 \\ DB^2 = \underline{a^2-2am+m^2} & 2CD^2 = \underline{\qquad 2m^2} \\ AD^2+DB^2=\underline{\underline{2a^2 \qquad +2m^2}}=2AC^2+2CD^2= & \underline{\underline{2a^2+2m^2}} \end{array}$$

SCH.—1. The Proposition may be expressed, "*the sum of the squares of any two lines is equal to twice the square of half their sum together with twice the square of half their difference.*"

Because $AD^2=(AC+CD)^2 \qquad =AC^2+2AC\cdot CD+CD^2$
And $BD^2=(BC-CD)^2=(AC-CD)^2=AC^2-2AC\cdot CD+CD^2$
By addition $AD^2+BD^2 \qquad =2AC^2 \quad + \quad 2CD^2$
But $AC=\frac{AD+DB}{2}$ and $CD=\frac{AD-DB}{2}$

Therefore, *the sum of the squares of any two lines*, &c. Q.E.D.

2. Or, *the sum of the squares is equal to half the square of the sum together with half the square of the difference;* thus $AD^2+BD^2=\frac{(AD+DB)^2}{2}+\frac{(AD-DB)^2}{2}$; or in numbers 14^2+6^2 *i. e.*, $196+36=\frac{20\times 20}{2}+\frac{8\times 8}{2}$ *i. e.*, $200+32=232$.

PROP. 10.—THEOR.

If a st. line be bisected and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line and of the square of the line made up of the half and the part produced.

CONS.—11. I., 3. I., 31. I., Pst. 1.

DEM.—29. I., Ax. 9.

Ax. 12. If a st. line meet two st. lines, so as to make the two int. angles on the same side of it taken together less than two rt. angles, these two st. lines, being continually produced, shall at length meet on that side on which the angles are less than two rt. angles.

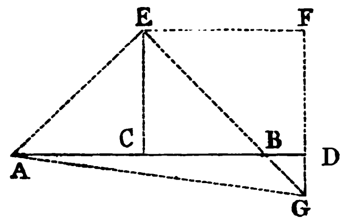
5. I., 32. I.

15. I. If two st. lines cut one another, the opp. or vertical angles shall be equal.

6. I., 34. I.

36. I. Parallelograms upon equal bases and between the same parallels are equal.

47. I.



Exp.	1	Hyp.	Let AB be bisected in C & produced to D;
	2	Concl.	then the squares on AD and DB = twice ($AC^2 + CD^2$).
Cons.	1	11. I.	From C draw CE at rt. angles to AB;
	2	3.I. & Pst. 1.	make CE = AC or CB, and join AE, EB;
	3	31. I.	through E and D draw EF ‖ AB, DF ‖ CE;
	4	Pst. 2 & 1.	produce FD and EB to meet in G, and join AG;
	5	Concl.	then $AD^2 + DG^2$ = twice ($AC^2 + CD^2$).
Dem.	1	C. 3 & 29. I.	∵ EF meets the ‖s EC and FD, the ∠s CEF and EFD = two rt. angles;
	2	Ax. 9.	and ∴ the ∠s BEF & EFD < two rt. angles.
	3	Ax. 12, Pst. 1.	the lines EB and FD ∴ will meet as in G. Join AG,
	4	C. 1 & 5. I.	then, ∵ AC = CE, the ∠ AEC = ∠ CAE;
	5	C. 1.	and ACE is a rt. angle;
	6	32. I.	∴ each ∠ AEC, CAE = half a rt. angle:
	7	Sim.	And ∠s CEB, CBE being each half a rt. angle, AEB is a rt. angle.
	8	D. 7, & 15. I.	And EBC being half a rt. ∠, its vertical ∠ DBG is half a rt. ∠ :
	9	29. I. & C. 1.	but ∠ BDG = the alt. ∠ DCE, which is a rt. ∠ ;
	10	32. I.	∴ the rem. ∠ DGB = half a rt. ∠ and also ∠ DBG;
	11	6. I.	and ∴ the side BD = the side DG.
	12	D. 10 & 34. I.	Again, ∵ ∠ EGF = half a rt. ∠, and ∠ EFG = ECD a rt. ∠ ;
	13	32. I. & D. 12	∴ rem. ∠ FEG = half a rt. ∠, and ∠ FEG = ∠ EGF;
	14	6. I.	and ∴ the side FG = the side FE.

DEM.	15	C. 1 & 36. I.	And ∵ EC = AC, ∴ the squares on EC and AC are equal,
	16	Ax. 1 & 6.	and ∴ the squares on EC and AC = twice the square on AC:
	17	47. I. & D. 16	but the square on AE = the two squares on EC and AC;
			and ∴ the square on AE = twice the square on AC.
	18	D. 14 & 36. I.	Again, ∵ FG = FE, FG sq. = FE squared;
	19	Ax. 1 & 6.	and ∴ the squares on FG and FE = twice the square on FE:
	20	47. I. & 34. I.	but the square on EG = the two squares on FG and FE;
			and ∴ the square on EG = twice the square on FE or on CD:
	21	D. 17.	And from the above, the square on AE = twice the square on AC;
	22	D. 20 & 21.	∴ $AE^2 + EG^2$ = twice $(AC^2 + CD^2)$.
	23	47. I.	But $AG^2 = AE^2 + EG^2$;
	24	Ax. 1.	∴ AG^2 = twice $(AC^2 + CD^2)$:
	25	47. I. & D. 11	but AC^2 also $= (AD^2 + GD^2) = (AD^2 + BD^2)$;
	26	Concl.	∴ $AD^2 + BD^2$ = twice $(AC^2 + CD^2)$.
	27	Recap.	*Wherefore, if a st. line be bisected and produced, &c.* Q.E.D.

Alg. & Arith. Hyp.—Let $AB = 2a = 20$; $\frac{AB}{2} = AC = CB = a = 10$; and $BD = m = 2$.

Alg. Then $AB + BD = AD = 2a + m$
and $CB + BD = CD = a + m$
therefore $(2a + m)^2 = 4a^2 + 4am + m^2$
$(+m^2)$ and $(2a + m)^2 + m^2 = 4a^2 + 4am + 2m^2$
Again $(a + m)^2 = a^2 + 2am + m^2$
$(+a^2)$ and $(a + m)^2 + a^2 = 2a^2 + 2am + m^2$
$(\times 2)$ and $2(a + m)^2 + 2a^2 = 4a^2 + 4am + 2m^2$
But $(2a + m)^2 + m^2 = 4a^2 + 4am + 2m^2$
therefore $(2a + m)^2 + m^2 = 2a^2 + 2(a + m)^2$

Arith. Then $AB + BD = AD = 20 + 2 = 22$
and $CB + BD = CD = 10 + 2 = 12$
therefore $(20 + 2)^2 = (4 \times 100) + (4 \times 20) + 4$
$(+4)$ and $(20 + 2)^2 + 4 = (4 \times 100) + (4 \times 20) + (2 \times 4)$
Again $(10 + 2)^2 = 100 + 40 + 4$

$(+100)$ and $(10+2)^2+100=200+40+4$
$(\times 2)$ and $2\,(10+2)^2+200=400+80+8$
But $(20+2)^2+4=400+80+8$
therefore $(20+2)^2+4=200+2\,(10+2)^2$
or $484+4=200+288=488$

In another form the Algebraical and Arithmetical illustration may be stated.

Let AC or $CB=a=10$; $BD=m=2$; $CD=a+m=10+2=12$; and $AD=2a+m=20+2=22$

$$\begin{array}{lll}
\text{Then } AD^2=4a^2+4am+m^2 & \text{And } 2AC^2 &= 2a^2 \\
\underline{DB^2= \qquad\qquad\quad m^2} & 2CD^2 &= \underline{2a^2+4am+2m^2} \\
AD^2+DB^2=4a^2+4am+2m & = 2AC^2+2CD^2 &= 4a^2+4am+2m^2
\end{array}$$

$$\begin{array}{rcr}
\text{Or, } 22\times 22=484 & & \text{And } 2\times 10\times 10=200 \\
2\times 2=\underline{4} & & 2\times 12\times 12=\underline{288} \\
\text{Sum } \underline{488} & = & \underline{488}
\end{array}$$

SCH.—1. Propositions 9 and 10 are applicable to Algebra, and like Prop. 5 and 6 are identical; for the different enunciations arise from the two ways of representing the differences between two lines.

GEN. USE AND APP. OF PROP. I.—X., BK. II.—These ten Propositions contain the whole theory of the relations of rectangles and squares formed by lines and their parts. As we have shown, all these relations may be expressed Algebraically, and may also be applied equally well to numbers.

"When lines are expressed numerically, various problems may be proposed respecting them, the solution of which may be derived from the preceding propositions. We shall here subjoin some of those problems, which will probably be sufficient to familiarize the student with such investigations."

1°. *Given the sum* S, *and difference* D, *of two magnitudes,* AB *the greater and* CD *the less; to find the magnitudes themselves.*—The formulas are,— $\frac{S}{2}+\frac{D}{2}=AB$ the greater; and $\frac{S}{2}-\frac{D}{2}=CD$ the less.

2°. Since the area of a rectangle expressed in numbers is equal to the product of its sides,—*if the Area be divided by one side, the quotient will give the other:* thus, let the sides be AB and BC; the rectangle is expressed by $AB\cdot BC$; and $\frac{AB\cdot BC}{BC}=AB$; $\frac{AB\cdot BC}{AB}=BC$; or let the area in numbers be 144, and the sides 9 and 16; then $\frac{144}{9}=16$; and $\frac{144}{16}=9$.

3°. "There are five quantities depending on a rectangle, any two of which being given, the sides of the rectangle can be found;

1°. The sum of the sides. 2°. The difference of the sides. 3°. The area. 4°. The sum of the squares of the sides. 5°. The difference of the squares of the sides.

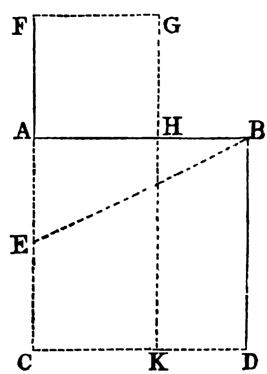
These being combined, 1° and 2°; 1° and 3°; 1° and 4°; 1° and 5°; 2° and 3°; 2° and 4°; 2° and 5°; 3° and 4°; 3° and 5°; and 4° and 5°, produce ten Problems,—of which the combination 3° and 5° alone presents any difficulty. The solutions will form useful exercises for the learner."—LARDNER'S *Euclid*, pp. 79 and 80.

PROP. 11.—PROB.

To divide a given line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

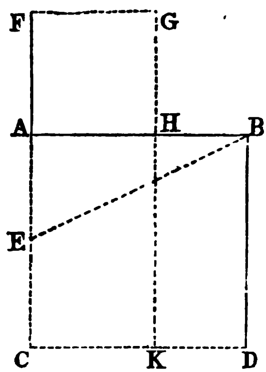
SOL.—46. I. To describe a square on a given line.
10. I. To bisect a given finite st. line.
3. I. From the greater line to cut off a part equal to the less.
Psts. 1 and 2.

DEM.—6. II. If a st. line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square of half the line bisected, is equal to the square of the st. line which is made up of the half and of the part produced.
47. I. In any rt. angled triangle, the square of the hypotenuse is equal to the sum of the squares of the base and perpendicular.
Ax. 3 and 1.
Def. 30. I. A square has all its sides equal, and its angles rt. angles.



EXP.	1	Datum.	Let AB be the given line;
	2	Quæs.	to divide it into two parts, so that rect. AB·BH = AH sqre.
CONS.	1	46. I.	On AB desc. a square ABDC;
	2	10.I.&Pst.1	bisect AC in E, making AE = EC, & join EB;
	3	Pst. 2, & 3.I.	produce CA to F, making EF = EB;
	4	46. I.	and on AF desc. the square AHFG:
	5	Pst. 2.	Produce GH to K:
	6	Sol.	the line AB is divided in H, so that AB·BH = the square on AH.
DEM.	1	C. 2 & 3.	∵ AC is bisected in E, and produced so that EF = EB;
	2	6. II. C. 2.	∴ □ CF·AF + AE^2 = EF^2 and EF^2 = EB^2;
	3	47. I.	also the square on EB = the squares on AB and AE:

DEM.	4	Sub. Ax. 3.	Take away AE^2 from each; then $\square\, CF.AF = AB^2$.
	5	Const.	But $\square\, FK = CF \cdot FA$, and $AD^2 = AB^2$;
	6	Ax. 1.	$\therefore$ the square $AD =$ the rect. FK:
	7	Sub. Ax. 3.	Take away AK, and the square $FH =$ the rect. HD:
	8	C. & Def. 30	But $\square\, HD$ is contained by $HB \cdot BD$, or $HB \cdot AB$;
	9	C. 4.	and FH is the square on AH;
	10	Concl.	$\therefore$ the $\square\, AB \cdot BH = AH^2$.
	11	Recap.	*Wherefore, the given line has been divided,* &c. Q.E.F.



N.B. A line thus divided is (in 30. vi.) said to be cut in extreme and mean ratio.

The Algebraical Solution may be given thus:

Take $AB = a$,—*required the point* H, *so that* $AB \cdot BH$ *shall equal* AH^2.

Let $AH = x$; then $HB = a - x$

Now by the prob. $a(a-x) = x^2$; or $x^2 = a^2 - ax$; *i. e.*, $x^2 + ax = a^2$

Solving the quadratic, we have $x^2 + ax + \frac{a^2}{4} = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}$

extracting the root, $x + \frac{a}{2} = \frac{a\sqrt{5}}{2}$. Hence $x = \frac{\pm a\sqrt{5} - a}{2}$

Thus AH the one part $= x = \frac{\sqrt{5}-1}{2} \cdot AB$,

and HB the other part $= a - x = a - AH = \frac{3-\sqrt{5}}{2} \cdot AB$.

Or, "required how far a given line must be produced so that the rectangle contained by the given line and the line made up of the given line and the part produced, may be equal to the square of the part produced."—POTTS' *Euclid*, p. 73.

Or, "to find two lines having a given difference, such that the rectangle contained by the difference and one of them may be equal to the square of the other."

The Arithmetical Solution cannot be expressed in whole numbers; we can however approximate to the values of AH and HB by extending the root of

5 to any number of decimal places desired. Thus, supposing $AB=a=10$, $AH=x=\frac{\sqrt{5}-1}{2}\cdot a$ and $HB=a-x=\frac{3-\sqrt{5}}{2}\cdot a$

1st. $\frac{\sqrt{5}-1}{2}\cdot a=\frac{2\cdot236068-1}{2}\times 10=\frac{1\cdot236068}{2}\times 10=6\cdot18034=AH$;

2nd. $\frac{3-\sqrt{5}}{2}\cdot a=\frac{3-2\cdot236068}{2}\times 10=\frac{\cdot763932}{2}\times 10=3\cdot81966=HB$;

And $AH+HB=AB=6\cdot18034+3\cdot81966=10\cdot00000$.

COR. I. *To cut a line in extreme and mean ratio, it must first be produced in extreme and mean ratio;* that is, $CF\cdot FA$ must equal AB^2.

COR. II. When a line CF, or its equal, is cut in extreme and mean ratio, the rectangle $AC\cdot(AC-AF)$ is equal to the square of AH, or AF; or $AC\cdot HB=AH^2$.

Hence, if a line, CF, be cut in extreme and mean ratio, the greater segment, AC, will be cut in the same manner, by taking in it a part equal to the less, AF; and the less, AF, will be similarly cut, by taking in a part equal to the difference $(AC-AF)$ or HB; and so on.

COR. III. A line CF being cut in extreme and mean ratio, the rect. $AC\cdot AF$ under its segments $=AC^2-AH^2$, the difference between their squares; thus $10\times 6\cdot18034=100-38\cdot1966=61\cdot8034$.

SCH.—Let A be a line cut in extreme and mean ratio, G the greater segment, L the less, and D the difference: then 1. $A^2+L^2=3G^2$; 2. $(A+L)^2=5G^2$; 3. $A\cdot D=G\cdot L$; and 4. $L^2=G\cdot D$.

USE AND APP.—This 11th Prop. is applied in 10. iv., to the drawing of an isosceles triangle, of which each of the angles at the base is double of the third angle; and, in 30. vi., to the cutting of a line in extreme and mean ratio,—that is, so that the whole line is to the greater segment as the greater is to the less. The construction of pentagons in bk. iv. also depends on this problem, and of regular bodies, as described in bks. xiii. xiv. and xv., called also the Platonic Solids.

PROP. 12.—THEOR.—*(Important.)*

In obtuse angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of

the sides containing the obtuse angle by twice the rectangle contained by the side upon which when produced the perpendicular falls, and the st. line intercepted without the triangle between the perpendicular and the obtuse angle.

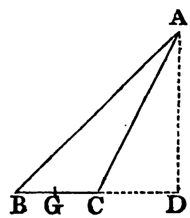
CONS.—12. I. To draw a st. line perpendicular to a given st. line of an unlimited length from a given point without it.

Pst. 2. A terminated st. line may be produced to any length in a st. line.

DEM.—4. II. If a st. line be divided into any two parts, the square of the whole line is equal to the square of the two parts, together with twice the rectangle contained by the parts.

Ax. 2. If equals be added to equals, the wholes are equal.

47. I. In any rt. angled triangle, the square described upon the side subtending the rt. angle is equal to the square described upon the side containing the right angle.



Ex. & CONS.	1	Hyp.	Let ABC a triangle have an obtuse $\angle$ ACB;
	2	12.I. & Pst.2	From A to BC produced draw a perp. AD;
	3	Concl.	then $AB^2 > (AC^2 + BC^2)$ by twice $BC \cdot CD$.
DEM.	1	Hyp.	For, since BD is divided into two parts at C;
	2	4. II.	$\therefore BD^2 = BC^2 + CD^2 + 2\,BC \cdot CD$:
	3	Add. & Ax.2	adding AD^2 to each, $BD^2 + AD^2 = BC^2 + CD^2 + AD^2 + 2\,BC \cdot CD$.
	4	C. 2 & 47. I.	But ADB being a rt. ang., $AB^2 = BD^2 + AD^2$;
	5	47. I.	and also $AC^2 = CD^2 + AD^2$;
	6	Concl.	therefore $AB^2 = BC^2 + AC^2 + 2\,BC \cdot CD$;
	7	Explan.	*i. e.*, $AB^2 > AC^2 + BC^2$ by twice $BC \cdot CD$.
	8	Recap.	*Wherefore, in obtuse angled triangles, if a perpendicular*, &c. Q.E.D.

Alg. & *Arith. Hyp.*—Let $BC = a = 5$; $CA = b = 6{\cdot}708$; $AB = c = 10$; and let $CD = m = 3$; $DA = n = 6$;

then $BD = a + m = 5 + 3 = 8$.

the triangles ABD and ACD are rt. angled, viz., at ADB and ADC.

Alg. Since $c^2 = (a+m)^2 + n^2$; and $b^2 = m^2 + n^2$;

therefore $c^2 - b^2 = (a+m)^2 - m^2 = a^2 + 2\,am + m^2 - m^2 = a^2 + 2\,am$,

and $c^2 = b^2 + a^2 + 2\,am$; *i. e.*, c^2 is greater than $b^2 + a^2$ by twice am.

Arith. Since $10^2=(5+3)^2+6^2=64+36$; & $6{\cdot}708^2$ or $45=3^2+6^2=9+36$; therefore $10^2-6{\cdot}708^2$, or $100-45=(5+3)^2-3^2=64-9=25+30$;
and $10^2=6{\cdot}708^2+5^2+(2\times5\times3)=45+25+30$:
i. e., 10^2 or 100 greater than $45+25$ by 30.

Use and App.—1. By this Proposition *the Area of a triangle may be ascertained when the three sides are known.*

Let the sides $AB=20$, $AC=13$, and $BC=11$.

Now (12. I.) $AB^2=AC^2+BC^2+2\,BC{\cdot}CD$;

therefore $AB^2-AC^2-BC^2$; or $AB^2-(AC^2+BC^2)=2\,BC{\cdot}CD$;

i. e., $2\,BC{\cdot}CD=400-(169+121)=400-290=110$;

therefore $BC{\cdot}CD=\frac{110}{2}=55$, and $\frac{BC{\cdot}CD}{BC}=CD=\frac{55}{11}=5$:

But (47. I.) $AC^2-CD^2=AD^2$; *i. e.*, $169-25=144$; and $AD=\sqrt{144}=12$.

We have now ascertained AD the altitude of the triangle ABC, and BC the base is given;

Again (41. I.) the Area of a triangle equals half that of a parallelogram on the same base and of the same altitude:

Thus the Area of triangle $ABC=\frac{AD{\cdot}BC}{2}=\frac{12\times11}{2}=\frac{132}{2}=66$.

Or, by bisecting the line BC in G, we obtain the same result; thus,

Since $\frac{AB^2-AC^2}{2}=BC{\cdot}DG$; therefore $\frac{BC{\cdot}DG}{BC}=DG$;

And $DG+\frac{BC}{2}=DB$; then (47. I.) $AB^2-BD^2=AD^2$; whence we find AD itself.

Numerically, $\frac{400-169}{2}=115{\cdot}5$, therefore $\frac{115{\cdot}5}{11}=10{\cdot}5=DG$;

And $10{\cdot}5+\frac{11}{2}=16$; then (47. I.) $400-256=144$; and $\sqrt{144}=AD$ as before.

Prop. 13.—Theor.

In every triangle, the square of the side subtending either of the acute angles is less than the squares of the sides containing that acute angle by twice the rectangle contained by either of these sides and the straight line intercepted between the perpendicular let fall upon it from the opposite angle and the acute angle.

Cons.—12. I., Pst. 2.

DEM.—7. II. If a st. line be divided into any two parts, the squares of the whole line and of one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Ax. 2, 47. I.

16. I. If one side of a triangle be produced, the ext. angle is greater than either of the int. and opposite angles.

12. II. In obtuse angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

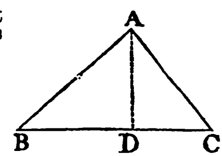
3. II. If a line be divided into any two parts, the rectangle contained by the whole and one of the parts equals the rectangle contained by the two parts together with the square of the aforesaid part.

Ax. 6. Doubles of the same are equal.

Ex. & Cons.	1	Hyp.	Let $\triangle$ ABC have the $\angle$ B acute,
	2	12. I.	On BC one of the sides let the perp. AD fall from $\angle$ BAC;
	3	Concl.	then $AC^2 < (AB^2+BC^2)$ by twice $BC \cdot BD$.

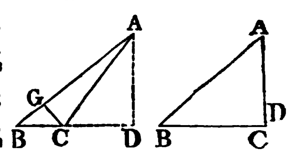
CASE I.—*Let the perp.* AD *fall within the triangle* ABC.

DEM.	1	C. 2 & 7. II.	Since BC is divided at D, $CB^2+BD^2=CD^2+2\,BC \cdot BD$;
	2	Add. & Ax. 2	On adding AD^2, $CB^2+BD^2+AD^2=CD^2+AD^2+2\,BC \cdot BD$:
	3	C. 2 & 47. I.	But $\angle$ s at D being rt. $\angle$ s $AB^2=BD^2+AD^2$, and $AC^2=CD^2+AD^2$;
	4	D. 2 & 3.	$\therefore CB^2+AB^2=AC^2+2\,BC \cdot BD$;
	5	Concl.	that is, $AC^2 < (BC^2+AB^2)$ by twice $BC \cdot BD$.



CASE II.—*Let the perp.* AD *fall without the* $\triangle$ ABC.

DEM.	1	C. & 16. I.	Since D is a rt. $\angle$, $\angle$ ACB > a rt. $\angle$.
	2	12. II.	$\therefore AB^2=AC^2+BC^2$ + twice $BC \cdot CD$;
	3	Add. Ax. 2.	add BC^2 to each, and AB^2+BC^2 $=AC^2$ + twice $(BC^2+BC.CD)$.
	4	C. & 3. II.	But BD being divided in C, the $\square$ BD.BC $=BC^2+BC.CD$;



DEM.	5	Ax. 6.	Now the doubles of equals being equal,
	6	Ax. 6.	twice BD · BC = twice BC · CD + twice BC^2 :
	7	D. 3 & 6.	∵ $AB^2 + BC^2 = AC^2$ + twice BD·BC ;
	8	Concl.	∴ AC^2 alone < $AB^2 + BC^2$ by 2 BD·BC.

CASE III.—Lastly, *let the side* AC *be perpendicular to* BC.

DEM.	1	47. I.	Here $AB^2 = AC^2 + BC^2$;
	2	Add. & Ax. 2	and adding BC^2, then $AB^2 + BC^2 = AC^2 + 2\ BC^2$, or 2 BC·BC.
	3	Recap.	*Wherefore, in any triangle, the square subtending,* &c. Q.E.D.

CASE I.—*Alg. & Arith. Hyp.* Let $BC=a=10$; $AB=c=\sqrt{61}=7{\cdot}8102$; $AC=b=\sqrt{41}=6{\cdot}4031$; $BD=m=6$; $AD=n=5$; and $DC=a—m=10—6=4$.

Alg. (47. I.) $c^2=n^2+m^2$ and $b^2=n^2+(a—m)^2$
Subtracting, $c^2—b^2=m^2—(a—m)^2=m^2—(a^2—2am+m)=2am—a^2$
Transpose, $a^2+c^2=b^2+2am$; or $b^2+2am=a^2+c^2$
therefore, b^2 is less than a^2+c^2 by $2am$.

Arith. (47. I.) $61=25+36$, and $41=25+16$
Subtracting, $61—41=(2\times10\times6)—100=120—100=20$
Transposing, $100+61=41+120$; or $41+120=100+61$
therefore, 41 is less than 100+61 by 120.

CASE II.—The perpendicular AD here passes out of the triangle, and the positions of C and D are changed; so that we have $BC=a=2$; $AB=c=\sqrt{61}$; $AC=b=\sqrt{41}$; $BD=m=6$; $AD=n=5$; and $DC=m—a=6—2=4$.

Alg. (47. I.) $c^2=m^2+n^2$ and $b^2=(m—a)^2+n^2$
Subtracting $c^2—b^2=m^2—(m—a)^2=m^2—m^2+2am—a^2=2am—a^2$
Transpose $a^2+c^2=b^2+2am$; or $b^2+2am=a^2+c^2$
therefore b^2 is less than a^2+c^2 by twice am

Arith. (47. I.) $61=36+25$, and $41=16+25$
Subtracting $61—41=36—16=36—36+24—4=(2\times12)—4$
Transpose $4+61=41+(2\times12)$; or $41+24=4+61$
therefore 41 is less than 4+61 by 24.

CASE III.—The perpendicular AD and the side AC coincide, and the points D and C; so that we have $BC=a=4$; $AD=AC=b=5$; and $AB=c=\sqrt{41}$.

Alg. (47. I.) $b^2+a^2=c^2$
Add a^2 and $b^2+2a^2=c^2+a^2$
i. e., b^2 is less than c^2+a^2 by $2a^2$ or $2aa$

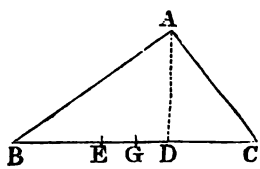
Arith. (47. I.) $25+16=41$
Add 16 and $25+32=41+16$
i. e., 25 is less than $41+16$ by 2×16

COR. I.—*If in the figure to* CASE II. *a perpendicular* CG *be drawn from the angle* C *to* AB, *the rectangle under the side* AB *and the part* GB *intercepted between the perpendicular and* B *is equal to the rectangle of* BC·DB. The two rectangles AB·BG and BC·DB each equal half the difference between the square AC and the squares AB and BC; *i.e.*, $\frac{(AB^2+BC^2)-AC^2}{2}$ $=BC\cdot DB=AB\cdot BG$. As $\frac{(61+4-41)}{2}=\frac{65-41}{2}=12=2\times 6=$ $7\cdot 8012\times 1\cdot 5364$.

SCH.—The Propositions 12 and 13 are of high importance,—for they contain the elements of Trigonometrical Analysis, or the Arithmetic of Sines.

USE AND APP.—1. In finding the Area of a Triangle, it is of the greatest advantage to obtain the perpendicular either by calculation or by measurement. *When the three sides of a triangle are given, the perpendicular may be obtained*, on the foregoing principles, in either of the ways following:

FIRST METHOD.—*When the perpendicular*, AD, *falls within the base.*

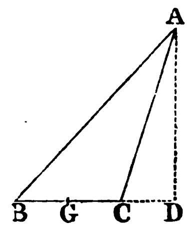


By 12. I., from ∠ A draw AD perpen. to BC;
47. I., in triangle ABD, $AB^2=AD^2+BD^2$
47. I., & in triang. ACD, $AC^2=AD^2+CD^2$
Subtract, and $AB^2-AC^2=BD^2-CD^2$
By 3 & 10. I. From BC cut off BE equal to DC, and bisect ED in G;
ED is evidently bisected in G, so that $2GD=ED$.

Now, since, by Cor. 5. II., the difference of the squares of two lines equals the rectangle under their sum and difference;
$AB^2-AC^2=(BA+AC)\ (BA-AC)=(BD+DC)\ (BD-DC)=BC\times 2GD=2BC\times GD$.
Dividing now both sides of the equation by $2BC$, we have
$GD=\frac{(BA+AC)\ (BA-AC)}{2BC}$; and since $BG=\frac{BC}{2}\therefore BD=\frac{BC}{2}+GD$,
and $AD=\sqrt{AB^2-BD^2}$ the perpendicular required.

EX. *Given* BC=5, AB=4, *and* AC=3; *required the perpendicular* AD.
Bisect BC in G; then $GD=\frac{7\times 1}{2\times 5}=\frac{7}{10}$; and $BD=\frac{5}{2}+\frac{7}{10}=3\frac{1}{5}$
therefore, by 47. I., $AD=\sqrt{16-\frac{256}{25}}=\sqrt{\frac{400}{25}-\frac{256}{25}}=\sqrt{\frac{144}{25}}=\frac{12}{5}=2\frac{2}{5}$

SECOND METHOD.—*When the perpendicular*, AD, *falls without the base.*



By 12. I., from $\angle$ A draw AD perpendicular to BC produced;

47. I., in triangle ABD, $AB^2=AD^2+BD^2$

47. I., and in triangle ACD, $AC^2=AD^2+CD^2$

Subtract, and $AB^2—AC^2=BD^2—CD^2$

By 10. I., let BC be bisected in G; then $BD+CD=2GD$, because $BD=2GB+CD$, and $BD+2CD=2GB+2CD=2GD$. Also $BD—CD=BC$.

But, by Cor. 5. II., $AB^2—AC^2=(AB+AC)(AB—AC)=(BD+CD)(BD—CD)$

Thus $BC\times 2GD$, or $2BC\times GD=(AB+AC)(AB—AC)$

Divide by 2 BC,—and $GD=\frac{(AB+AC)(AB—AC)}{2BC}$

Now $CD=GD—GC$; And (47. I.) $AD=\sqrt{AC^2—(GD—GC)^2}$

Ex. *Given the three sides* BA=5, AC=4, *and* BC=2; *required the perp.* AD.

By the formula $GD=\frac{(5+4)(5—4)}{2\times 2}=\frac{9}{4}=2\frac{1}{4}$

$$CD=2\tfrac{1}{4}—1=1\tfrac{1}{4} \text{ and } AD=\sqrt{16—\frac{25}{16}}=\frac{\sqrt{256—25}}{16}=\frac{\sqrt{231}}{4}$$

$$=\frac{15\cdot198684}{4}=3\cdot799671.$$

2. When the intercepts between the foot of the perpendicular and the angles on the base are ascertained, the *Area of the Triangle* is by the First Method $\frac{BD+DC}{2}\times AD$; and by the Second Method $\frac{BD—DC}{2}\times AD$; that is, half the rectangle of the base multiplied by the perpendicular.

PROP. 14.—PROB.

To describe a square that shall be equal to a given rectilineal figure.

SOL.—45. I. To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle.
Def. 30. A square, &c.
Psts. 2 and 3; 3. I. 10. I. To bisect a given finite st. line.
Pst. 1.

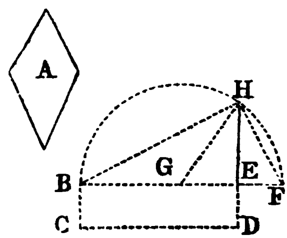
DEM.—5. II. If a st. line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts together with the square of the line between the points of section equals the square of half the line.

Def. 15 and 16. A circle is a plane figure contained by one line called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. And this point is called the centre of the circle.

47. I. The square of the hypotenuse, &c.

Ax. 3. If equals be taken, &c.

Ax. 1. Things equal, &c.



Exp.	1	Datum.	Let A be the given rectil. figure,
	2	Quæs.	to make a square equal to the fig. A.
Cons.	1	45. I.	Describe a right angled ▭ BCDE = A:
	2	Def. 30.	If the side BE = ED the fig. is a square;
	3	Sup.	but if BE $\neq$ ED,
	4	Pst. 2 & 3. I.	produce BE to F, and make EF = ED:
	5	10. I. Pst. 3.	Bisect BF in G, and from G with GB describe the semicircle BHF;
	6	Pst. 2 & 1.	produce DE to H, and join GH:
	7	Sol.	the square on EH = the figure A.
Dem.	1	C. 5.	$\because$ BF is bisected in G, and cut unequally in E;
	2	5. II. Def. 16	$\therefore BE \cdot EF + EG^2 = GF^2$, and $GF^2 = GH^2$:
	3	47. I.	But the square on $GH = EH^2 + EG^2$;
	4	Sub. & Ax. 3	taking away EG^2, then $BE \cdot EF = EH^2$:
	5	C. 1. Ax. 1.	But ▭ $BD = BE \cdot ED$, or $BE \cdot EF$;
	6	Concl.	therefore ▭ BD = the square on EH;
	7	C. 1 & D. 6.	But ▭ BD = the fig. A: $\therefore EH^2 = A$.
	8	Recap.	*Wherefore, a square has been made equal*, &c.

Q.E.F.

Alg. & Arith. Hyp.—Given the fig. $A = ab = 36$, in area; required the side $EH = x$ of a square equal in area. Let the lines, $EH = x$; BE one side of the rectangle $= b = 9$, and ED the other side $= a = 4$.

Alg. By the problem, $ab = x^2$ (therefore, ($\sqrt{}$) $\sqrt{ab} = x$ the side of the square.	*Arith.* $4 \times 9 = x^2$ and $\sqrt{36} = x = 6$ the side of the square.

USE AND APP.—1. By this proposition we may find a *mean proportional to two given lines*, as demonstrated in Prop. 13, bk. vi.

Given the lines BE and EF; their sum $=2r=13$; BE $=x=9$; EF $=2r-x=13-9=4$: required to find y, a mean proportional to x, and $2r-x$; *i. e.*, so that $x : y :: y : 2r-x$.

Bisect BE in G; $GF=r=6\cdot5$	$(9\times4(+6\cdot25=6\cdot5^2=42\cdot25$
By 5. II. $x\cdot(2r-x)+(x-r)^2=r^2$	$42\cdot25=y^2+6\cdot25$
But $GF^2=r^2=y^2+(x-r)^2$	$9\times4=36=y^2$
Subtract $(x-r)^2$ And $x\cdot(2r-x)$ $=y^2$	

Extract the $\sqrt{}$; $y=\sqrt{36}=6$ the mean proportional between 9 and 4.

Briefly, ***Multiply the two numbers, and take the square root of their product*** for the mean proportional.

2. The same Prop. 14 serves also for *approximating* to the square of curve-lined figures, and even of the circle itself; for the circle may be regarded (as in 41. I., Use 4) as a polygon with an infinite number of sides;—and if we reduce this polygon to a square, we *nearly* obtain the square of the circle.

3. The Areas of all plane figures are calculated in squares; and by means of reducing any right figure, first into a rectangle, and then into a square, we obtain the ready means of finding the Area in the units of surface.

N.B. A selection of the most useful and remarkable problems and theorems, which may be inferred from the principles developed in the second book, will be found in the APPENDIX to the Gradations in Euclid, SERIES II., *Problems and Theorems*, bk. ii.

REMARKS ON BOOK II.

1. Of the fourteen Propositions of this book, the *ten* first contain the theory of the relations of the rectangles and squares on divided lines; the *twelfth* and *thirteenth* the theory of the relation between the square of any one side of a triangle, and the squares of the other two sides, whatever be the angles,—and constitute the foundation of the Arithmetic of Sines.

2. Lines cut into *any two parts* are considered in Propositions 2, 3, 4, 7, and 8.

3. Lines cut into *two equal* and *two unequal parts*,—in Propositions 5, 6, 9, and 10.

SYNOPSIS OF BOOK II.

CASE I.—*Given two lines, as* A *undivided* = 10, *and* BC = 60 *divided into the parts* BD = 30, DE = 20, *and* EC = 10.

A ——

B ——— D ——— E — C

PROP. I. $A.BC = A.BD + A.DE + A.EC$; or $10 \times 60 = (10 \times 30) + (10 \times 20) + (10 \times 10) = 600$.

Cor. $2\,A.\frac{1}{2}BC$; or $3\,A.\frac{1}{3}BC$; or $4\,A.\frac{1}{4}BC$, &c. $= A.BC$.
20×30; 30×20; $40 \times 15 = 10 \times 60 = 600$.

CASE II.—*Let a line be divided into any two parts;* AB = 10, *representing the whole line;* AD = 7, *and* DB = 3, *the unequal segments.*

A ——— D ——— B

PROP. II. $AB.AD + AB.DB = AB^2$; or $(10 \times 7) + (10 \times 3) = 10 \times 10 = 100$.

PROP. III. $AB.BD = AD.DB + DB^2$; or $(10 \times 3) = (7 \times 3) + (3 \times 3) = 30$.

Cor. 1. $AB^2 - DB^2 = (AB + DB)(AB - DB)$; or $100 - 9 = 13 \times 7 = 91$.

2. $AD^2 - DB^2 > (AD - DB)^2$ by $2\,DB.(AD - DB)$; or $49 - 9 > 16$ by $2 \times 3 \times 4 = 24$.

PROP. IV. $AB^2 = AD^2 + DB^2 + 2\,AD.DB$; or $100 = 49 + 9 + (2 \times 7 \times 3) = 49 + 9 + 42$.

Cor. 1. Parallelograms about the diam. of a square are also squares.

2. $AB^2 = 4\left(\frac{AB}{2}\right)^2$; or $100 = 4 \times 25$.

3. $\frac{AB^2}{2} = 2\left(\frac{AB}{2}\right)^2$; or $\frac{100}{2} = 2 \times 25$.

4. When a line is divided into any number of parts, as $x = 5$; $y = 3$; $z = 2$;
then $AB^2 = x^2 + y^2 + z^2 + 2(x.y + x.z + y.z)$
or, $100 = 25 + 9 + 4 + 2(15 + 10 + 6)$

PROP. VII. $AB^2+BD^2=2\,AB.BD+AD^2$, or $100+9=(2\times10\times3)+49=109$.

Cor. 1. $AB^2+BD^2=2\,AB.BD+(AB—BD)^2$; or $100+9=60+(10—3)^2=109$.

2. $(AB+BD)^2$; (AB^2+BD^2); and $(AB—BD)^2$ are in Arithmetical progression, the common difference being $2\,AB.BD$; or,

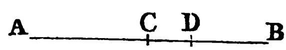
169, 109, and 49; the com. dif. being $2\times10\times3=60$.

PROP. VIII. $4\,AB.BD+AD^2=(AB+BD)^2$; or $4\,(10\times3)+49=(10\times3)^2=169$.

Cor. 1. $(AD+DB)^2=4\,AD.DB+(AD—DB)^2$; or $(7+3)^2=4\,(7\times3)+(7-3)^2=100$.

2. $4\left(\frac{AD+DB}{2}\right)^2=4\,AD.DB+4\left(\frac{AD—DB}{2}\right)^2$; or $4\times25=(4\times21)+(4\times4)=84+16$.

CASE III.—*Let a line be divided equally and unequally,* AB = 10 *representing the whole line;* AC *or* CB = 5, *the half line; and* AD = 7; DB = 3, *the unequal parts.*



PROP. V. $AD.DB+CD^2=CB^2$; or $(7\times3)+(2\times2)=5\times5$; *i.e.*, $21+4=25$.

N.B. The arith. mean $=\frac{AD+DB}{2}=AC$; the com. dif. $=\frac{AD—DB}{2}=CD$.

$\therefore\left(\frac{AD+DB}{2}\right)^2=AD.DB+\left(\frac{AD—DB}{2}\right)^2$; or $25=21+4$.

Or, Let AD and DB be regarded as two independent lines; then $AD.DB+\left(\frac{AD—DB}{2}\right)^2=\left(\frac{AD+DB}{2}\right)^2$; or $(7\times3)+4=\left(\frac{7+3}{2}\right)^2=25$.

Again, $(AD+DB).(AD—DB)+DB^2=AD^2$; or $(10\times4)+9=7\times7=49$.

or, $AD^2—DB^2=(AD+DB).(AD—DB$; or $49—9=10\times4=40$.

Cor. $AC^2—CD^2=(AC+CD)\,(AC—CD)$; or $25—4=7\times3=21$.

LARDNER'S *Cor.* 1. The rectangle is a *maximum* when AB is bisected by D; its *maximum* value $=\left(\frac{AB}{2}\right)^2$

2. The sum of the squares of the parts is a *minimum* when AB is bisected: the *minimum* value being $2\left(\frac{AB}{2}\right)^2$
3. Of all rectangles having the same perimeter, the square contains the greatest area.
4. Of all rectangles equal in area, the square is contained by the least perimeter.
5. If a perpendicular be drawn from the vertex of a triangle, as in Prop. 13, to the base $(AB + AC).(AB \sim AC) = (BD + DC).(DC \sim BD)$,—or $(7 \cdot 8102 + 6.4031) \times (7.8102—6.4031) = (6 + 4) \times (6 - 4) = 20$.
6. The difference between the squares of the sides of a triangle, as in Prop. 13, is equal to twice the rectangle under the base and the distance of the perpendicular from the middle point, x, of BC: *i.e.*, $AB^2—AC^2 = 2\ BC.Dx$; or $61—41 = 2 \times 10 \times 1 = 20$; *or*, $61—41 = (2 \times 2) \times 5 = 20$.

N.B.—If the perpendicular AD falls within the base,

$$Dx = \frac{BD \sim DC}{2};\ \text{and}\ BC = BD + DC:$$

but if the perp. AD falls without the base,

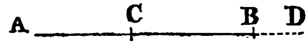
$$Dx = \frac{BD + DC}{2};\ \text{and}\ BC = BD—DC.$$

PROP. IX. $AD^2 + DB^2 = 2\,(AC^2 + CD^2)$; or $49 + 9 = 2\,(25 + 4) = 58$.

Or $AD^2 + DB^2 = 2\left(\frac{AD+DB}{2}\right)^2 + 2\left(\frac{AD—DB}{2}\right)^2$; or $(2 \times 25) + (2 \times 4) = 58$.

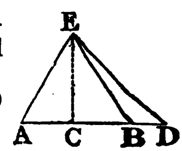
Or $AD^2 + DB^2 = \frac{(AD+DB)^2}{2} + \frac{(AD—DB)^2}{2}$; or $\frac{100}{2} + \frac{16}{2} = 50 + 8$.

CASE IV.—*Let a line be bisected and produced*, $AB = 10$ *representing the original line;* AC *or* $CB = 5$, *its half; and* $BD = 3$ *the part produced.*



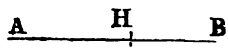
PROP. VI. $AD.DB + CB^2 = CD^2$; or $(13 \times 3) + (5 \times 5) = 64$.

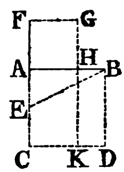
Cor. Let $AB = 10$, the base of an isosceles triangle, be bisected in C, and produced to D; join $ED = 11.7898$; then $ED^2—EB^2 = AD.DB$; or $139—100 = 13 \times 3 = 39$.



PROP. X. $AD^2 + DB^2 = 2\,(AC^2 + CD^2)$; or $169 + 9 = 2 \times (25 + 64) = 2 \times 89 = 178$.

CASE V.—*A line is cut in extreme and mean ratio, when the rectangle of the whole line and one segment equals the square of the other segment.*





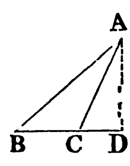
PROP. XI. AB so cut in H that $AB.HB = AH^2$
$10 \times 3.81966 = 38.1966 = 6.18034 \times 6.18034$.

Cor. 1. To cut a line in extreme and mean ratio, it must first be produced in extreme and mean ratio; *i. e.*, in fig. Prop. 11, $CF.FA$ must equal AB^2.

2. When a line CF, or its equal, is cut in extreme and mean ratio, the rectangle $AC.(AC—AF) = AH^2$ or AF^2; or $AC.HB = AH^2$.

3. A line CF being cut in extreme and mean ratio, $AC.AF = AC^2—AH^2$; *i. e.*, $10 \times 6.18034 = 100—38.1966 = 61.8034$.

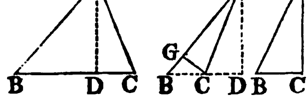
CASE VI.—*The measure of the square of the side of an obtuse-angled triangle opposite to the obtuse angle.*



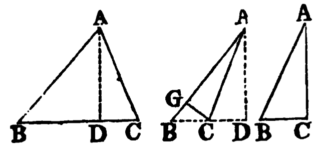
PROP. XII. $AB^2 = BC^2 + AC^2 + 2\,BC.CD$,
$100 = 25 + 45 + (2 \times 15)$.
or, $AB^2 > BC^2 + AC^2$ by $2\,BC.CD$,
$100 > 25 + 45$ by 30.

N.B. The Area of a triangle may be ascertained when the three sides are known; for Area $= \frac{AD \cdot BC}{2}$.

CASE VII.—*The measure of the square of the side subtending an acute angle.*



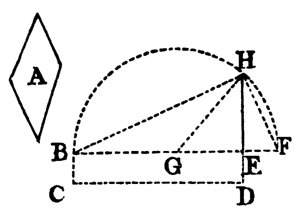
PROP. XIII. 1°. $CB^2 + AB^2 = AC^2 + 2\,BC.BD$; $100 + 61 = 41 + 120$.
or $AC^2 < (CB^2 + AB^2)$ by $2\,BC.BD$; $41 < 161$ by 120.



$2^{o.}$ $AB^2 + BC^2 = AC^2 + 2\,BD.BC$; $61 + 4 = 41 + (2 \times 12)$;
or $AC^2 < (AB^2 + BC^2)$ by $2\,BD.BC$; $41 < 65$ by 24.
$3^{o.}$ $AC^2 < (AB^2 + BC^2)$ by $2\,BC.BC$; $25 < 41 + 16$ by 2×16.

If in $2^{o.}$, a perpendicular CG be drawn from $\angle$ C to AB, the
$\square$ AB.GB=BC.DB; or $7.8012 \times 1.5364 = 12 = 2 \times 6$.

Case VIII.—*A square is found equal to a given rectilineal figure.*



Prop. XIV. By Const., $\square$ BCDE = A;
$BE.EF + EG^2 = GF^2 = GH^2$; $9 \times 4 + 6.25 = 42.25$;
$\square$ BD = BE.ED, or BE.EF; $9 \times 4 = 36$;
$\square$ $BD = EH^2 = A$. $\sqrt{36} = 6 =$ EH side of the square.

PRACTICAL RESULTS.

The Practical Results at which we arrive from studying the principles demonstrated in the first and second Books of Euclid, are among the most useful and important of any belonging to Geometry: when united to a few leading theorems of the third and sixth books, they constitute by far the most fruitful source of scientific progress; they are, in fact, the foundation on which our after reasonings depend respecting all magnitudes, space, and number,—the form and motions of the heavenly bodies,—and all the laws by which the material universe is governed.

Nearly all the Problems in Euclid's Six Books of Plane Geometry may be derived directly and immediately from the first and second books. Indeed, for the simple construction of Geometrical Figures, scarcely any principles are demanded which are not to be found in, or naturally inferred from, those books. More fully therefore to give the Learner an idea of the wide application of the knowledge he may have acquired, I will briefly exhibit the Practical Results to which I believe we have attained: they may be arranged under four leading divisions:—

1st. *The Problems for the Construction of Geometrical Figures both stated and demonstrated in Books* I. *and* II.;

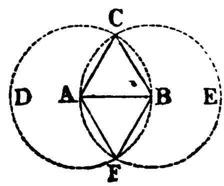
2nd. *Problems in the other books which, though demonstrated under their respective Propositions, are yet most intimately connected with the first and second, for their construction and solution;*

3rd. *Principles of Construction for Geometrical Instruments to measure lines and angles; and for Geometrical figures to exhibit the representative values of actual magnitude and space;* and

4th. *Principles which without requiring that we should measure all the boundaries of a Surface, yet enable us accurately to calculate distances, magnitudes, and areas.*

I.—The Problems for the Construction of Geometrical Figures both stated and demonstrated in Books I. and II.

Problem 1. *On a given line* AB *to describe an equilateral triangle.*

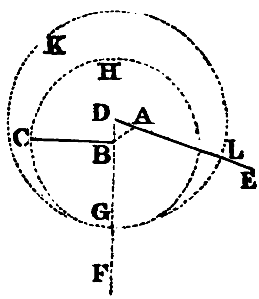


From the extremities of AB, with the distance AB for radius, describe two circles intersecting in C, and F; join the point C to the points A and B, and the required triangle is drawn. (1. I.)

Practically it is sufficient to draw the arcs intersecting in C, and to join the points A, B, and C.

An isosceles triangle may be drawn by taking the length of the equal sides, and using it as the radius from A and B.

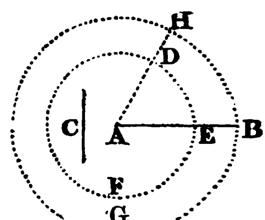
Prob. 2. *From a given point* A *to draw a line equal to a given line* CB.



Join A and B, and on AB construct an equil. triangle; produce its sides DA and DB indefinitely; from B, with the radius BC, describe a circle HCG; and from D, with DG, the circle GKL: AL is the line of the required length. (2. I.)

Practically, the distance CB will be transferred by the compasses or by some other instrument.

Prob. 3. *From the greater,* AB, *of two given lines to cut off a part equal to* C, *the less.*



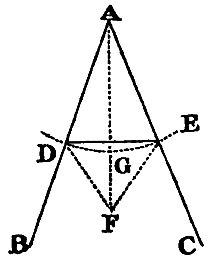
From A, with a radius equal to C, cut the line AB in E: AE will be equal to C. (3. I.)

A less line may be made equal to a greater, by describing an arc with the radius of the greater, and by producing the less until it touches the circumference.

Prob. 4. *To bisect a given rectilineal angle,* BAC; *that is, to divide it into two equal parts.*

From the angular point A, describe an arc, cutting the sides AB and AC in D and E; from D and E, with the same radius, draw arcs intersecting in F: on joining A and F, the angle BAC is bisected. (9. I.)

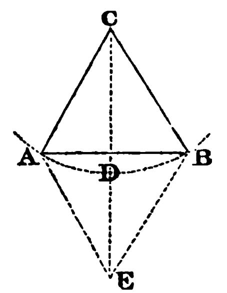
By following the same method the bisected parts may also be further bisected. Practically, the division of an angle into any of the powers of 2 will be effected by measuring the angle BAC with a protractor, and dividing the angular distance by the required power of 2; as, 4, 8, 16, &c.



Prob. 5. *To bisect a given finite straight line,* AB, *and to continue the bisection of the bisected parts.*

From A and B, with the same radius, describe arcs intersecting both above and below the line in C and E; join CE: and the line AB is bisected in D. Follow the same method with AD or DB, and for all the subsequent bisections. (10. I.)

As in the angle, the division by 2, or by any power of 2, will be performed practically by measuring the given line, and by dividing the lineal measurement by 2, or the required power of 2, 4, 8, 16, &c.

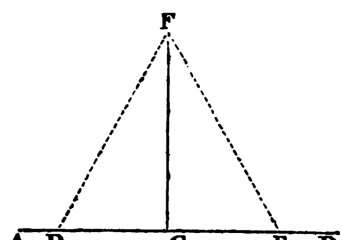


Prob. 6. *To draw a perpendicular,* i. e, *a straight line at right angles to a given straight line* AB, *from a given point* C, *in the same.*

By 3. I., take CD equal to CE; and from D and E, with the same radius, describe arcs intersecting in F; the st. line from F to C is the perpendicular required. (11. I.)

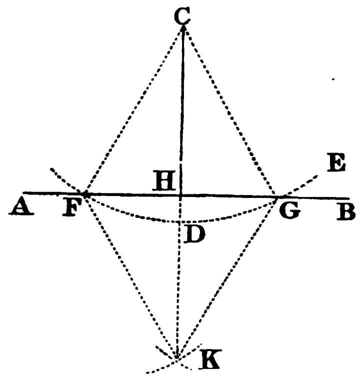
The drawing of a perpendicular from the extremity of a line is shewn in § II., Prob. 5, Prac. Res.

The practical method is by means of an instrument called a square; or by placing a protractor with its edge along the given line AB, and its centre at C, the angular point; and then joining C with 90° on the graduated edge of the protractor.



Prob. 7. *To draw a perpendicular to a given straight line,* AB, *of an unlimited length from a given point,* C, *without it.*

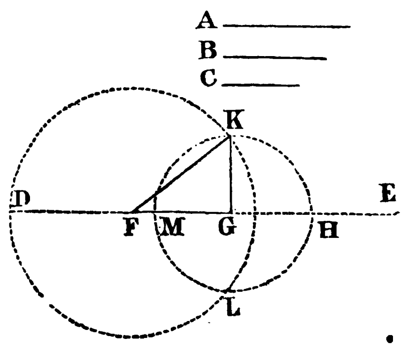
From C, with a radius CD, extending below or beyond the line AB, intersect AB in F and G; and from F and G, with the same radius, describe arcs intersecting in K; join CK: and CH is the perpendicular required. (12. I.)



The easiest way is to use the square, or the protractor: lay one edge of the square along the given line, and slide the square, keeping that edge on the given line, until the other edge just covers the given point: — a line along the edge will be the perpendicular. The protractor is used in a similar way.

PROB. 8. *To make a triangle of which the sides shall be equal to three given straight lines,* A, B, *and* C, *but of which any two whatever must be greater than the third.*

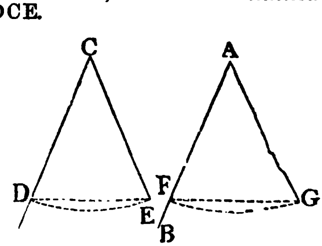
Draw a line of an indefinite length DE, and on DE, from D, in succession lay lines, by 3. I., DF equal to A, FG to B, and GH to C: from F, with radius FD, describe a circle: and from G, with radius GH, another circle; join their point of intersection K, with F and G; and the figure FGK is the triangle required. (22. I.)



In the same way one triangle may be made equal to another. In practice, a line FG is taken equal to B; and from F and G, arcs, with radii equal to A and C, are drawn intersecting in K, and K joined to F and G.

PROB. 9. *At a given point* A, *in a given line* AB, *to make a rectilineal angle equal to a given rectilineal angle,* DCE.

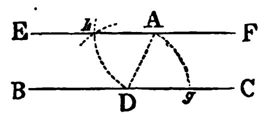
From C and A, with the same radius, describe arcs DE and FG; take DE as a radius, and with it from F describe another arc intersecting arc FG in G; join GA; and BAC is the angle required. (23. I.)



Or, the given angle DCE may be measured by the protractor; and by placing the centre of the protractor at the given point A, with its edge along AB, and noting the same number of degrees

from AB as equal the measurement of angle DCE, and joining that note or mark with A, the required angle will be drawn. The same may also be done by the line of chords.

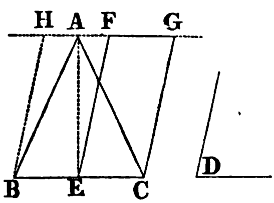
PROB. 10. *To draw a straight line through a given point* A, *parallel to a given straight line*, BC.



Join A and any point D, in BC; from D with DA describe the arc A*g*; and from A, with the same radius AD, describe the arc D*h*; by 23. I., make the angle DAE equal to the angle ADC: the line EA will be parallel to the given line BC. (31. I.)

In practice, the parallel ruler, or the triangular square, is used.

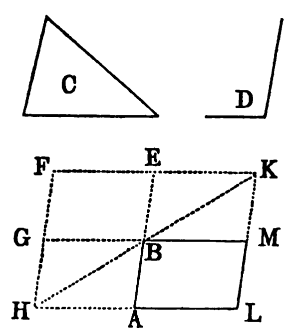
PROB. 11. *To describe a parallelogram that shall be equal to a given triangle* ABC, *and have one of its angles equal to a given rectilineal angle* D.



By 10. I. bisect BC in E; by 23. I. at E make an angle CEF equal to D; through C and A, by 31. I., draw CG parallel to EF, and AG parallel to BC: the figure FECG is the parallelogram required. (42. I.)

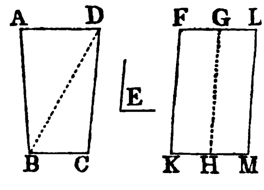
If it were required *to describe a triangle equal to a given parallelogram* FECG, *and having an angle equal to a given angle* D, the method would be,—Produce the parallels GF and CE; and, by 3. I., make EB equal to EC; and by 23. I. make the angle CBA equal to D; join CA;—and the triangle ABC will equal the parallelogram FECG, and have an angle equal to D.

PROB. 12. *To a given line*, AB, *to apply a parallelogram which shall be equal to a given triangle* C, *and have one of its angles equal to a given angle* D.



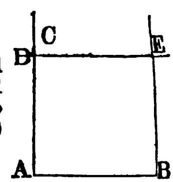
By 42. I. make a parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D; produce EB; and by 3. I. make BA equal to the given line; produce FG to H, and, by 31. I., through A draw HL parallel to GB or FE; join HB and produce it until it meets FE produced, in K; through K draw KL parallel to EA or FH; and produce GB to M: the figure ABML is the parallelogram required. (44. I.)

PROB. 13. *To describe a parallelogram equal to a given rectilineal figure* ABCD, *and having an angle equal to a given rectilineal angle* E.



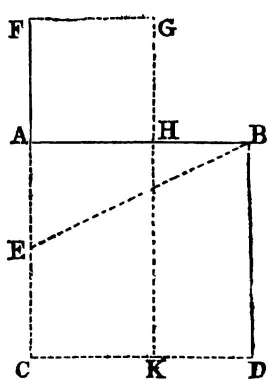
Divide the given figure into triangles, and draw an indefinite line KM; and at K, by 23. I., make the angle FKM equal to angle E: to FK apply, by 42. I., a parallelogram FH equal to the triangle ADB; and to GH the parallelogram GM equal to the triangle DBC: the whole figure FKML having an angle K equal to E, is a parallelogram equal in area to the rectilineal figure ABCD. (45. I.)

PROB. 14. *To describe a square on a given straight line,* AB.



From A, by 11. I., draw the perpendicular AC, and on it make, by 3. I., AD equal to AB; through B and D, by 31. I., draw the parallels, BE to AD, and DE to AB: the figure ABED is the square required. (46. I.)

PROB. 15. *To divide a given line,* AB, *into two parts, so that the rectangle contained by the whole line and one of its parts,* AB·BH, *shall be equal to the square of the other part,* AH.

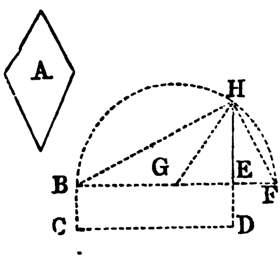


On AB, by 46. I., describe the square AD; bisect, 10. I.. CA in E, and join EB; produce CA indefinitely, and, by 3. I., make EF equal to EB; on AF, by 46. I., construct the square FH, and produce GH to K: the line AB is divided at H, so that AB·BH equals the square on AH. (11. II.)

N.B. In Prop. 30, bk. vi., a line thus divided is said to be cut *in extreme and mean ratio; i. e.,* so that the whole line AB shall be to the greater segment AH, as the greater segment AH to the less segment HB.

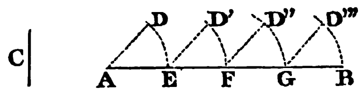
PROB. 16. *To describe a square that shall be equal to a given rectilineal figure* A.

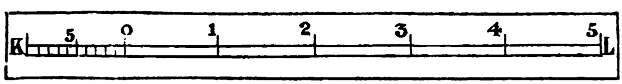
Describe, by 45. I., a rectangle BCDE equal in area to the given figure A: if the sides BE and ED are equal, the construction is finished; but if not, produce BE until by 3. I. EF equals ED; by 10. I. bisect BF in G, and with GB as radius, describe the semicircle BHF; produce DE to meet the semicircle in H; the square on EH is the square equal to the rectangle BD and to the figure A. (14. II.)



Subsidiary Problems.—Books I. and II.

Prob. 17. *To construct a Scale of Equal Parts.*





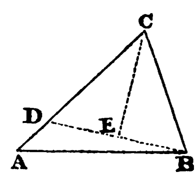
Take a st. line AB of indefinite length towards B; and let C be the given st. line that is to be cut off from AB: from AB, by 3. I., cut off a part equal to C, as AE; then again from EB another part equal to C, as EF; and so on: the parts in AB are each equal to C and to one another; and AB is a scale of equal parts. (3. I.)

On the same principle the line KL is divided. Each of the parts numbered 1, 2, 3, 4, is representative of 10 equal parts, and those between 0 and K of units; or if the smaller divisions be tenths, the larger will be units.

By such a scale the comparative lengths of lines are ascertained.

Prob. 18. *Given* AB *the base, angle* A *the less angle at the base, and* AD *the difference of the two sides of a triangle, to construct it.*

On AD the line forming with AB the angle A, by 3. I., set the difference AD of the two sides; join DB; and, by 10 and 11. I., bisect DB by the perpendicular EC; and produce EC and AD to meet in C: ABC is the triangle required. (19 and 4. I.)



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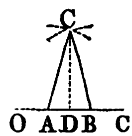
PROB. 19. *On a given line* AB, *to describe an isosceles triangle, having each of the equal sides double of the base.*

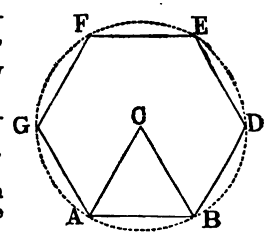
Produce AB both ways; and by 3. I., make BC, AO each equal to AB: OB and AC each equals twice AB. With AC from A, and with BO from B, describe arcs cutting in C; join C to A and B: ABC is the isosceles triangle with sides each the double of the base. (22. I.)



PROB. 20. *To describe an isosceles triangle, the base* AB *being given, and* F *a line equal to each of the sides.*

Bisect the base in D, and raise a perpendicular at D; from A with radius equal to F, draw an arc cutting the perpendicular in C; join CA and CB: the required isosceles triangle is ACB. (22. I.)

PROB. 21. *To construct a regular polygon,* as ABDEFG.

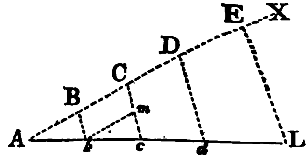


By 32. I., the formula for the angular magnitude of each angle in a regular polygon is, $\angle A = \frac{(S \times 180) - 360}{S}$, the angle between any two sides; $\angle C = \frac{360}{S}$, for the angle at the centre between two radii to the angular points. (*Note,* 32. I., p. 108.)

1°. *When the side* AB *is not given.* With any radius, as CA, describe a circle; at C the centre, make an angle equal to $\frac{360}{S}$, and produce CB to the circumference; join AB: the line AB will, if set round the circumference, divide it into as many parts as there are sides to the polygon; join the points, and the polygon is made.

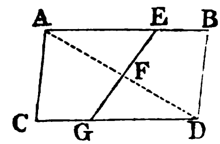
2°. *When the side* AB *is given.* By formula, $\angle A = \frac{(S \times 180) - 360}{S}$; ascertain the angles GAB, ABD, and construct them; by 9. I., bisect each of the angles; and the lines of bisection, AC, BC, intersect in C: from C, with radius CA, describe a circle; if now, with AB as radius, successive arcs be cut off from the circumference, there will be as many arcs as sides, and the drawing of the chords to those arcs will complete the polygon. (*Use and App.* 32. I., p. 108.)

PROB. 22. *To divide a finite straight line* AL, *into any given number of equal parts, as four.*



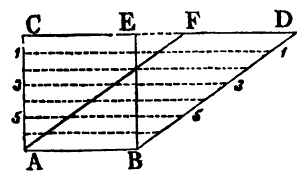
At A, draw a line of indefinite length, AX, making with AL the angle XAL; take a line AB on AX; and, by 3. I., set along AX three other lines BC, CD, DE each equal to AB; join EL; and through the points D, C, and B, by 31. I., draw Dd, Cc, Bb, each parallel to EL: the line AL will be divided into four equal parts in the points b, c, d. (*Use and App.* 34. I., p. 111.

PROB. 23. *From a point* E *in the side* AB *of a parallelogram, to divide the parallelogram into two equal portions.*



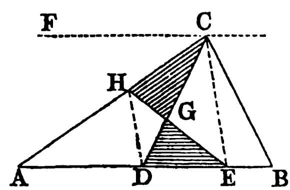
Draw the diagonal AD, and, by 10. I., bisect it in F; join EF, and produce EF to G: the line EG makes the portion AEGC equal to the portion EGDB. (*Use and App.* 6, 34. I., p. 113.)

PROB. 24. *To convert a parallelogram,* ABDF, *into a rectangle,* ABCE, *of equal area.*



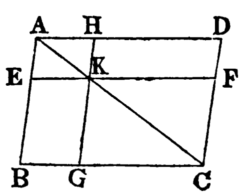
Produce indefinitely the parallel DF; at B and A, the extremities of the other parallel AB, by 11. I., raise the perpendiculars BE and AC to meet the parallel DC; then ABCE is a rectangle equal in area to the parallelogram ABDF. (*Use and App.* 35. I., p. 115.)

PROB. 25. *Given a triangle,* ABC, *and a point,* H, *in one side,* AC; *from that point to bisect the triangle.*



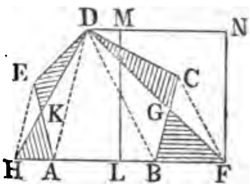
Through C the vertex, by 31. I., draw CF parallel to AB; by 10. I. bisect AB in D, and join DC; join also DH; and from C draw CE parallel to HD, and join HE: then the triangle AHE will equal the trapezium CHEB. (*Use and App.* 2, 38. I., p. 120.)

PROB. 26. *On a given line,* EK, *to draw a parallelogram equal to a given parallelogram,* KD.



Produce FK, and by 3. I. make EK equal to the given line; produce indefinitely the sides DF, HK, and DH; and by 31. I. draw through E, AB parallel to HG or DC; draw the diagonal AK, and produce it until it cuts DF produced in C; through C draw a parallel to DA or FE, meeting HK and AE produced in G and B: the parallelogram BK will be equal to the given parallelogram KD. (*Use and App.* 43. I., p. 127.)

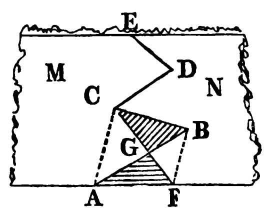
PROB. 27. *To change any right-lined figure,* ABCDE, *first into a triangle, and then into a rectangle of equal area.*



Divide the figure into triangles by joining DA, DB; produce AB indefinitely; through E, by 31. I., draw EH parallel to DA, and through C, CF par. to DB; join DH and DF: then the triangle DHF is equal in area to ABCDE.

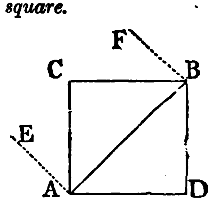
Next, draw DN parallel to HF; by 10. I. bisect HF in L; at L, 11. I., raise the perpendicular LM, and draw FN parallel to LM: the fig. LMNF is a rectangle equal in area to DHF, which is equal to ABCDE. (*Use and App.* 2, 45. I., p. 132.)

PROB. 28. *To make straight a crooked boundary,* ABCDE, *between two fields,* M *and* N.



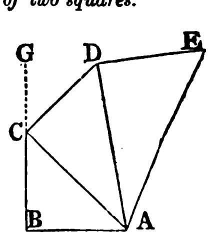
Draw AC, the subtend to angle B; and through B, 31. I., FB parallel to AC, and join CF: the crooked boundary AB, BC is now converted into the single boundary CF. In a similar way FC and CD will be converted into one boundary; and this last and DE into a single boundary: and thus the crooked boundaries AB, BC, CD, DE, will be changed into one straight boundary without affecting the areas of the two fields. (*Use and App.* 3, 45. I., p. 132.)

PROB. 29. *Given the diagonal,* AB, *to construct a square.*



At A and B, by 11. I., draw the perpendiculars AE, BF; by 9. I., bisect the right angles by AC and BC meeting in C; and by 31. I., draw through A and B, parallels, AD to BC, and BD to AC: then the figure ABCD is the square required. (46. I., p. 133.)

PROB. 30. *To find a square equal to the sum of any number of given squares, as on* AB, BC, CD, DE; *or a square that is a multiple of any given square,* AB; *or a square that is equal to the difference of two squares.*



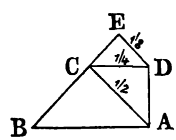
1°. Set the lines AB, BC, representative of the two squares on AB, BC, at right angles, and join AC; then $AC^2 = AB^2 + BC^2$; at C place CD, representative of the square on CD, at right angles to AC; and $AD^2 = AC^2 + CD^2$: and at D, forming a right angle with AD, place DE representative of the square on DE; join AE: and the square on AE equals the squares on DE, DC, CB, and BA. (*Cor.* 3, 47. I., p. 137.)

2°. Supposing AB to be representative of the line on which the given square is constructed, its multiple square will be obtained in a similar way;

for in this case, BC, CD, DE being each equal to AB, the square on AE is the multiple of the square on AB. (*Cor.* 3, 47. I., p. 137.)

3°. Let AB be the less line, and AC the greater; at B, the extremity of the less, raise a perpendicular BG, and from A at the other extremity, with AC as radius, inflect on BG the greater line: the square of the intercept CB will equal the difference of the squares on AC and AB. (*Cor.* 3, 47. I., p. 137.)

PROB. 31. *To make a square that shall be the half, fourth, &c., of a given square on* AB.

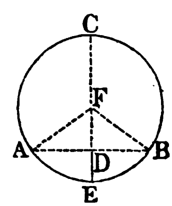


At A and B make the angles each equal to half a right angle; C being a right angle, the square on AC will be one-half of the square on AB. Again, at A and C make the angles CAD, ACD each equal to half a right angle: then the square on CD will be half that on AC, or one-fourth of the square on AB. By a similar process a square may be obtained that is $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c., of the original square. (*Cor.* 3, 47. I., p. 138.)

II.—The Problems in Books 3, 4, and 6, which though demonstrated under their respective Propositions, are, for their Construction and Solution, most intimately connected with the First and Second Books.

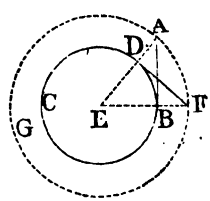
Book III.

PROB. 1. *In a given circle,* ABC, *to find the centre.*



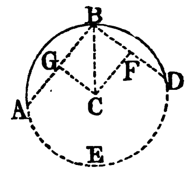
Draw any chord AB; and, 10. I., bisect it at D; from D, 11. I., raise a perpendicular produced both ways to meet the circumference in C and E; bisect CE in F; and F is the centre of the circle. (1. III.)

PROB. 2. *To draw a tangent to a given circle* BCD, *from a given point, either without as* A, *or in the circumference as* D.



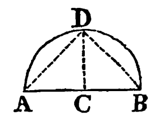
Find the centre E of the circle BCD, and join AE; from E with EA describe the circle AFG; from D, 11. I., draw DF at right angles to EA, and join EBF and AB: then AB shall touch the circle in B, and DF in D. (17. III.)

PROB. 3. *To describe a circle through three given points,* ABD, *not in the same line; or to complete the circle of which a segment,* ABD, *is given.*



Join A, B, and B, D; 10 and 11. I., bisect AB and BD by perpendiculars GC, FC, meeting in C: C is the centre of the circle, and a radius from C to either A, B, or D, will be the means of completing the circle. (25. III.)

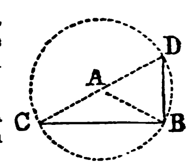
PROB. 4. *To bisect a given circumference,* ADB.



Join the extremities of the arc A, B; 10 and 11. I., and bisect AB by a perpendicular from C: the line CD will bisect the given circumference in the point D. (30. III.)

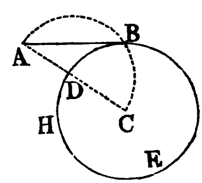
LEMMA.—31. III. The angle in a semicircle is a right angle.

PROB. 5. *To draw a perpendicular,* 1°. *from a point* B *in a line* CB, *or at its extremity;* 2°. *from a point* D, *out of a line; and,* 3°. *so as to be a tangent to a given circle* HBE.



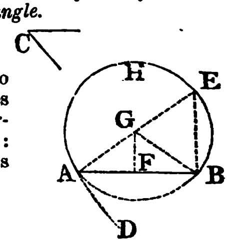
1°. Take any point A, above the line, and with a radius equal to AB describe an arc to cut the given line in C; join C and A, and produce CA to meet the arc in D: DB is the required perpendicular.

2°. Join D and C; and, 10. I., bisect CD in A; and from A with AC describe the semicircle CBD, and join DB: DB is the perpendicular.



3°. Join the given point A, out of the circle, and C the centre of the circle; 10. I., bisect AC in D; and with DA describe a semicircle: where the semicircle cuts the circle, B, is the tangent point, and AB the tangent from A. *Or,* from B, a point in the circle, draw a line BC to the centre; and at B, 11. I., draw a right angle: BA is the tangent from B.

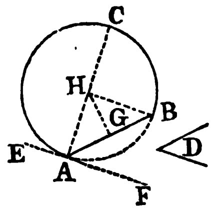
PROB. 6. *Upon a given straight line* AB, *to describe a segment of a circle containing an angle equal to* C, *a given rectilineal angle.*



At A, 23. I., make the angle BAD equal to angle C; and, 11. I., at A make AE at right angles with DA; 10 and 11. I., bisect AB in F by the perpendicular GF; with GA or GB describe a circle: and in the segment AHB the angle AEB equals BAD, which also equals angle C. (33. III.)

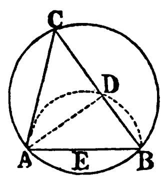
PROB. 7.—*Given the angle* D *equal to the vertical angle of a triangle, and the base* AB, *to find the* locus *of the vertex.*

At A, by 23. I., make the angle BAF equal to angle D, and, 11. I., angle FAH equal to a right angle; bisect, 10 and 11. I., AB in G by the perpendicular GH; and from H, with radius HA, or HB describe a circle ABC: the *locus* of the vertex will be at any point in the arc of the segment ACB. (33. III.)



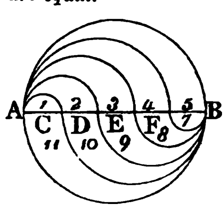
PROB. 8. *Given* AB *the base, angle* ACD *the vertical angle, and* AD *the perpendicular from the extremity of the base* A *on the opposite side* BC, *to construct the triangle.*

Make on AB, by Prob. 6, a segment containing the given angle ACD; bisect AB, 10. I., in E; and with rad. EA or EB describe the semicircle ADB; and from A, inflect the perpendicular AD upon the semicircle in the point D: BDA is a right angle, 31. III., and BD produced to C, and CA joined, give the triangle required. (33. III.)



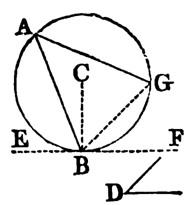
PROB. 9. *To divide a given circle of which the diameter is* AB, *into any number of equal parts, of which the perimeters also are equal.*

By Use and App., 34. I., divide the diameter AB into the required equal parts at C, D, E, F; then on one side, from A describe the semicircles 1, 2, 3, 4, 5, &c.; and on the other side from B, the semicircles 7, 8, 9, 10, 11, &c.,—of which the diameters are BF, BE, BD, BC; so shall the parts 1 and 11, 2 and 10, 3 and 9, 4 and 8, and 5 and 7, be equal each to the other, both in Area and Perimeter.—LESLIE'S *Geometry.*



PROB. 10. *From a given circle,* ABG, *to cut off a segment which shall contain an angle equal to a given rectilineal angle* D.

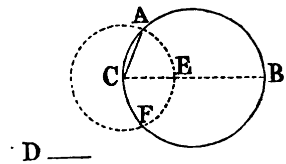
Draw any radius CB; and at B, 11. I., draw EF making right angles with CB; at B, 23. I., make the angle FBG equal to angle D: the segment BAG contains an angle BAG equal to the angle GBF, which equals angle D. (34. III.)



Book IV.

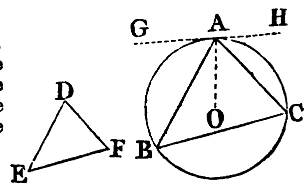
Prob. 1. *In a given circle,* ABC, *to place a st. line equal to a given st. line* D, *not greater than the diameter of the circle.*

Draw BC the diameter of ABC: 3. I., make CE equal to D; and with CE describe the circle AEF: CA is the line required. (1. IV.)



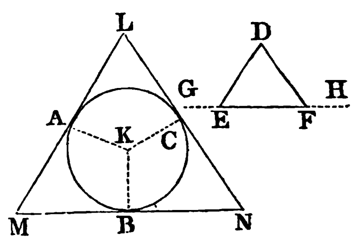
Prob. 2. *In a given circle,* ABC, *to inscribe a triangle equiangular to a given triangle,* DEF.

Draw OA any radius; and at A, 11. I., GH a tangent; also at A, 23. I., make the angles HAC and GAB equal to the angles DEF and DFE, and join BC: the triangle ABC is equiangular with the given triangle DEF. (2. IV.)



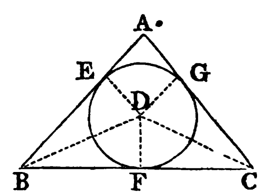
Prob. 3. *About a given circle,* ABC, *to describe a triangle equiangular to a given triangle,* DEF.

Produce EF both ways to G and H; find K the centre of the circle, and draw a radius KB; at K, 23. I., make the angle BKC equal to angle DFH, and angle BKA equal to angle DEG; at the points A, B, and C draw, 11. I., lines at right angles to AK, BK, CD, and produce them both ways until they meet: the fig. LMN is equiangular with the triangle DEF, and described about the circle ABD. (3. IV.)



Prob. 4. *In a given triangle,* ABC, *to inscribe a circle.*

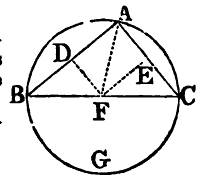
By lines BD, CD, meeting in D, bisect, 9. I. the angles B and C; from D, 12. I., drop perpendiculars DE, DF, DG, to AB, BC, and CA; with radius DE describe a circle: the circle EFG is inscribed in the given triangle. (4. IV.)



Prob. 5. *About a given triangle,* ABC, *to describe a circle.*

By 10 and 11. I., bisect AB and AC by perpendiculars meeting in F, and join FA; with FA as radius describe the circle ACGB; it is described about the given triangle. (5. IV.)

This is the same as Prob. 3, from bk. III., to describe a circle through three given points, A, B, and C.

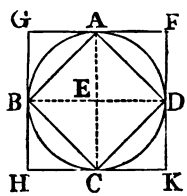


PROB. 6. *In a given circle,* ABCD, *to inscribe a square.*

Draw BD and AC two diameters at right angles to each other, and join A, B, C, and D: the figure ABCD is a square in the circle. (6. IV.)

PROB. 7. *About a given circle,* ABCD, *to describe a square.*

Draw two diameters, AC and BD, making right angles at E; at the extremities A, B, C, D, draw lines 11. I., at right angles to EA, EB, EC, and ED, and produce them until they meet in G, H, K, F: the fig. GHKF is a square about the circle. (7. IV.)

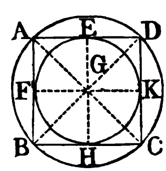


PROB. 8. *To inscribe a circle in a given square,* ABCD.

By 10 and 11 I., bisect each of the sides by the perpendiculars EG, FG, HG, KG, intersecting in G; and with GE as radius describe the circle EFHK in the square. (8. IV.)

PROB. 9. *To describe a circle about a given square,* ABCD.

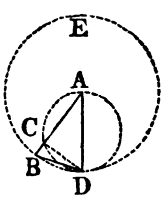
Draw the diagonals AC, BD, and from their point of intersection G, with GA, describe a circle: the circle ABCD is about the given square. (9. IV.)



PROB. 10. *To describe an isosceles triangle having each of the angles at the base double of the third angle.*

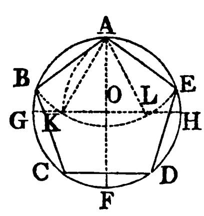
Take any line AB, and, by 11. II., divide it so that the rectangle AB·BC shall equal the square on CA; from A, with radius AB, describe the circle BDE, and in it, by 1. IV., place the line BD equal to AC; and join DA: ABD is the isosceles triangle required. (10. IV.)

For the proof, join DC, and about the triangle ADC, 3. III., describe the circle ACD.

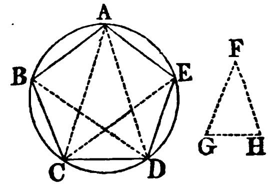


PROB. 11. *In a given circle,* ABCDE, *to inscribe an equilateral and equiangular pentagon.*

Draw two diameters, AF and GH, at right angles; 10. I., bisect the radius OH in L; and with AL as radius describe an arc cutting OG in K: the distance AK is equal to one side of the pentagon; and if set round the circle will complete the figure. If the arcs are bisected, and the points of bisection joined, a decagon will be drawn. (P. 10, bk. XIII.)

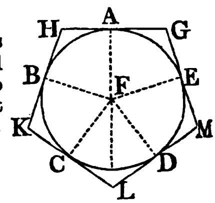


Or, 10. IV., describe an isosceles triangle, FGH, having each of the angles at the base double of that at the vertex; and, 2. IV., in the given circle inscribe a triangle ACD, equiangular with FGH; 9. I., bisect the angles ACD, ADC, and let the bisecting lines be produced to meet the circumference in B and E: the points A, B, C, D, E, are the angular points of the required pentagon. (11. IV.)



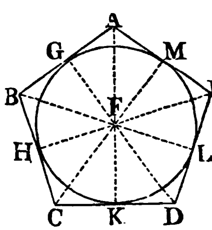
PROB. 12. *About a given circle,* ABCDE, *to describe an equilateral and equiangular pentagon.*

By 11. IV. let A, B, C, D, E, be the angular points of a pentagon inscribed in the circle ABCDE; and from F, the centre of the circle, let lines be drawn to those points; at A, B, C, D, and E draw lines, 11. I., at right angles to FA, FB, FC, FD, and FE: the fig. HGMLK is the pentagon about the given circle. (12. IV.)



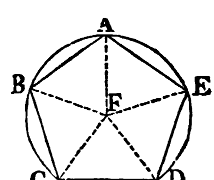
PROB. 13. *In a given equilateral and equiangular pentagon,* ABCDE, *to inscribe a circle.*

By 9. I. bisect the angles BCD, CDE, &c., by lines CF, DF, meeting in F; from F, 12. I., draw perpendiculars FH, FK, FL, to BC, CD, DE; and with any one, as FH, as radius, describe a circle: this circle GHKLM will be inscribed in the pentagon ABCDE. (13. IV.)



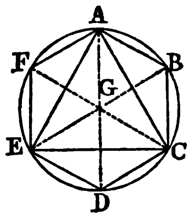
PROB. 14. *To describe a circle about a given equilateral and equiangular pentagon,* ABCDE.

By 9. I. bisect the angles ABC and BCD by the lines BF and CF; and from their intersection in F draw FD, FE, FA; with either of these as radius, describe a circle, as ABCDE: this circle will be about the given pentagon. (14. IV.)



PROB. 15. *To inscribe a regular hexagon in a given circle,* ABCDEF.

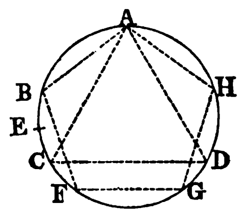
Draw a diameter of the circle, as AD, and bisect it in G; from A with a distance equal to the radius AG, draw arcs cutting the circle in F and B; and from D with the same radius, arcs cutting in E and C; join the points A, B, C, D, E, and F,—and the hexagon is inscribed in the circle. (15. IV.)



N.B. By joining the alternate points, A, C, E, an equilateral triangle ACE is inscribed; and by bisecting each of the arcs of the hexagon, and joining the points, a dodecagon is formed and inscribed.

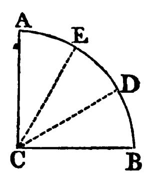
PROB. 16. *In a given circle,* ABCD, *to inscribe an equiangular and equilateral quindecagon.*

By 2 and 11. IV. inscribe in the circle an equilateral triangle ACD, and a regular pentagon; AB being a side of the pentagon, and AC a side of the triangle; 30. III., bisect BC in E, and BE or EC will be the fifteenth part of the circumference,—from which the quindecagon may be constructed. (16. IV.)



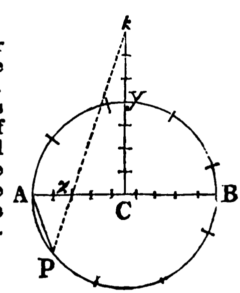
PROB. 17. *To trisect a quadrant.*

From A and B, with the radius of the circle, describe arcs cutting the quadrant in D and E; join EC, DC, and the quadrant is trisected.



PROB. 18. *In a given circle,* ABP, *to inscribe any regular polygon; or to divide the circumference of a circle into any assigned number of equal parts.*

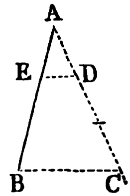
By Use and App. 34. I., Divide the diameter AB into the same number of equal parts as the figure has sides, suppose 9; from the centre C, 11. I., draw a perpendicular Ck; divide the radius Cy into four equal parts; and set off three of those parts from y to k; join k and z the second of the divisions from A, and produce kz to the circumferenee in P: the line AP will be the side of the required polygon.—*See a Treatise on Mensuration, Irish National Schools,* p. 19; *Demonstrations,* p. 53.



Book VI.

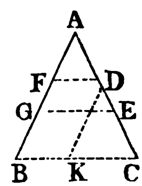
Lemma.—2. VI. If a straight line be drawn parallel to any side of a triangle, it divides the other sides, or those sides produced, into proportional segments.

Prob. 1. *To cut off from a given straight line,* AB, *any part required.*



At A place any line AC, making with AB the angle BAC; in AC take any point D, and in succession, by 3. I. cut off from AC as many parts each equal to AD as there are to be parts in AB; join BC; and, by 31. I., through D draw DE parallel to BC: whatever part AD is of AC, the same part is AE of AB. (9. VI.)

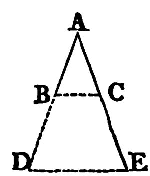
Prob. 2. *To divide a given straight line,* AB, *similarly to a given divided line,* AC.



Set AB and AC so as to make an angle BAC, and join BC; through D and E, the points of division on AC, by 31. I., draw DF and EG parallel to BC; in the points F and G, AB is similarly divided to AC.

N.B. When one line is cut similarly to another, the several segments of the one are proportional to the several segments of the other.

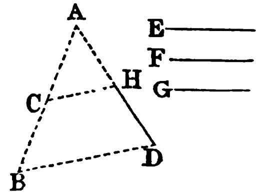
Prob. 3. *To find a third proportional to two given straight lines,* AB *and* AC.



Place the two lines so as to form an angle BAC; produce AC indefinitely, and AB so that BD, 3. I., is equal to AC; and join BC; through D, 31. I., draw DE parallel to BC: CE is the third proportional; *i. e.*, AB : AC :: AC : CE. (11 VI.)

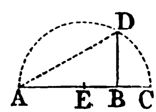
N.B. A repetition of the same construction will avail for continuing the progression.

Prob. 4. *To find a fourth proportional to three given straight lines,* E, F, *and* G.



Draw any two lines from A so as to make the angle BAD; by 3. I., on AB make AC equal to E, CB equal to F, and AH equal to G; and, 31. I., through B draw BD parallel to CH: HD is the fourth proportional. (12. VI.)

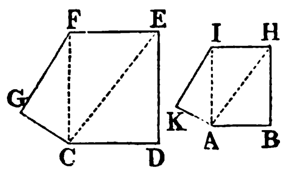
PROB. 5. *To find a mean proportional between two given straight lines,* AB *and* AC.



Set the given lines so as to form one straight line AC; by 11. I., bisect AC in E, and, with EA or EC as radius, describe a semicircle; at B, by 12. I., raise a perpendicular BD to D: DB is the mean proportional; *i.e.*, AB : BD :: BD to BC. (13. VI.)

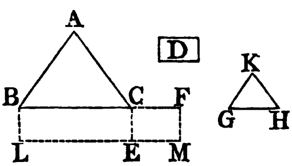
Or, let $AC=2r=13$; $AB=x=9$; and $BC=2r-x$: then $BD^2=y^2=2rx-x^2$.

PROB. 6. *On a given line,* AB, *to construct a rectilineal figure similar and similarly situated to a given rectilineal figure,* CDEFG.



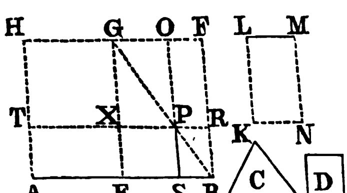
To divide the given figure into triangles, draw CE and CF; by 23. I., at B and A make the angles ABH, BAH, HAI, and IAK equal to the angles EDC, DCE, ECF and FCG; at H make angle AHI equal to angle CEF; and at I, angle AIK equal to angle CFG: the figure ABHIK is similar and similarly situated to the given figure CDEFG. (18. VI.)

PROB. 7. (The most extensive problem in the Elements of Geometry.) *To describe a rectilineal figure similar to one given rectil. figure,* ABC, *and equal to another rectil. figure,* D.



By Cor. 45. I., on BC describe the parallelogram BE equal the figure ABC; and on CE a parallelogram equal to D; and with the angle FCE equal to the angle CBL, by 13. VI., or 14. II., find GH a mean proportional between BC and CF; and on GH, describe by 18. VI., GHK similar to ABC: GHK is the figure required. (25. VI.)

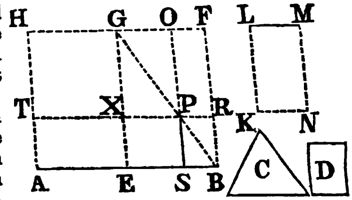
PROB. 8. *To a given line,* AB, *to apply a parallelogram equal to a given rectilineal figure* C, *and deficient by a figure similar to a given parallelogram* D. *But the rectilineal figure must not be greater than the parallelogram applied to half the given line, whose defect is similar to the given parallelogram* D.



By 10. I., bisect AB in E; and by 18. VI., on EB construct EBFG similar to D; and by 31. I., complete the parallelogram AG, making it either equal to C, or greater than C. If AG is equal to C, the problem is solved,—for on AB has been applied the parallelogram AG equivalent to C, and deficient by the parallelogram EF similar to D.

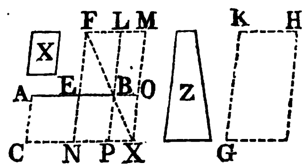
But if the area of AG be greater than that of C, by 25. VI., make the parallelogram KM equal to the excess of parallelogram EF above C, and

similar to D. Make XG, 3. I., equal to KL, and GO to LM; and complete the parallelogram GOPX; XO and EF have the same diagonal; draw that diagonal GPB, and produce XP to R, and OP to S, and complete SR: then the parallelogram TS equal to the given figure C is applied to the given line AB deficient by the parallelogram SR, which is similar to the given parallelogram D. (28. VI.)

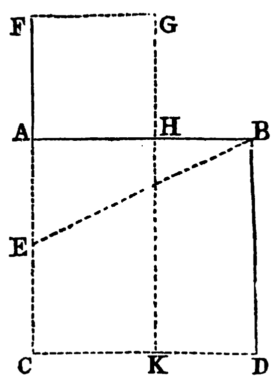


PROB. 9. *To a given straight line*, AB, *to apply a parallelogram equal to a given rectilineal figure* Z, *and exceeding by a parallelogram* BX *similar to another given parallelogram* X.

By 10. I., bisect AB in E; and by 18. VI. on EB construct EBLF similar to X, and KGH, (equalling parallelogram EL, and figure Z) similar to X. Produce FL and FE, making, 3. I., FLM equal to KH, and FEN to KG; complete the parallelogram MN: EL and MN have the same diagonal FX; draw that diagonal, and produce EB to O, LB to P, MO and NP to X: then the parallelogram AX equal to the given figure Z, and exceeding by the parallelogram PO which is similar to X, has been applied to the straight line AB. (29. VI.)



PROB. 10. *To cut a given straight line*, AB, *in extreme and mean ratio*, i.e., *so that the whole line is to the greater segment as the greater segment to the less.*

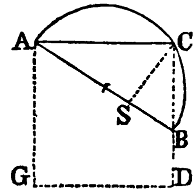


On AB, by 46. I., describe the square BC; and, by 29. VI., apply to AC the parallelogram CG equal to the square BC, and exceeding by the square AG similar to the square BC: the line AB is cut in H in extreme and mean ratio. (30. VI.)

Or, by 11. II., divide AB in H, so that AB·HB equals the square on AH: then also AB is cut in extreme and mean ratio in the point H.

PROB. 11. *To make a square approximating in area to the area of a given circle on the diameter* AB.

In Cor. 8. VI., it is proved that $AB \cdot AS = AC^2$. Divide AB the diameter into fourteen equal parts, and of these set off eleven from A to S; at S, by 11. I., raise the perpendicular SC terminated in the circumference; join AC, and the square on AC will closely approximate to the area of the circle of which AB is the diameter.—*Mensuration, Irish National Schools*, p. 20.



III.—Principles of Construction.

§ 1. For Geometrical Instruments to Measure Straight Lines, *i. e.*, Distances,—and Angles.

1. *Instruments for measuring st. lines* are constructed by following out the Applications given in pages 51 and 111, when treating of Propositions 3 and 34, bk. I.

Any st. line may be assumed as the unit: if it is the tenth of an inch, then, by the repetition of that tenth ten times, an instrument to measure an inch will be made; if an inch be the unit, and that inch be repeated twelve times, we have the measure for a foot; if a foot, and there be three repetitions of it, we have a yard; and if a yard be taken 1760 times, we have a mile. A very usual unit is the link, containing 7·92 inches, and 100 of these being taken, the Gunter's chain equal to 66 feet is formed.

The instruments commonly employed for the measurement of Lines, are the Inch, Foot, &c.; the Offset Staff and Chain; the Perambulator—a wheel of which the circumference is $\frac{1}{8}$ of a chain, or 8·25 feet; and Screws of various sizes. The threads on the screw are at exactly equal distances, and a single turn of the screw measures one of those equal distances. The spaces between the threads are generally small; if there are 100 threads in the lineal inch, each turn of the screw represents $\frac{1}{100}$ of an inch, and the hundredth part of a turn $\frac{1}{100}$ of $\frac{1}{100}$ or $\frac{1}{10000}$ of an inch.

2. *Instruments for measuring angles* are only in part constructed on strictly Geometrical principles. At p. 75, Cor. 1 and 2, 15. I., we find that if a circle be round a point, all the possible angles from the centre to the circumference taken together equal four right angles: the circumference being always divided into 360°, each quadrant contains 90°; the quadrant may be trisected,—$\frac{2}{3}$ of the quadrant, or 60°, measures the sextant; the 30° being bisected give two fifteens; but the division of 15° into *fifteen* equal parts can only be effected mechanically;—see p. 64, where it is shown that if the method of successive bisections had been followed, the degrees of the circle would be in strict accordance with the principles established in Plane Geometry.

The instruments generally employed for measuring angles, are the Cross Staff for right angles,—the Theodolite for any angle,—the Quadrant for an angle not exceeding 90°,—and the Sextant for an angle not exceeding 60°. The Goniometer is used for measuring the angles of crystals, &c.

and HALLEY'S REFLECTING QUADRANT for taking the angles between luminous points, as the stars, &c. Spirit levels and a plummet enable us to find a perpendicular, or a line at right angles to the horizontal line; and a Graduated or Geometrical Square fastened on a staff and placed in the true vertical position, to measure angles of elevation.

An angle may also be measured by any line of equal parts, by simply measuring the same distance along the lines that form the angle, and from the extremities of the distances thus measured, measuring the subtend to the angle. The Magnetic Compass, fixed on a circle called a Circumferenter, is often used for surveying mines or large tracts of land where much accuracy is not required.

The knowledge and use of such instruments are only to be attained by a practical acquaintance with them.

3. *There is a great difference between Lineal Magnitude and Angular Magnitude.* The unit in Lineal Magnitude is a fixed quantity,—three feet is always the same amount of space: but the unit in Angular Magnitude is a variable quantity, changing with the circumference of the circle. Whatever may be the size of the circle, its circumference is divided into 360 degrees; so that a degree may at one time represent the tenth of an inch,—at another time comprise ten million miles. This is familiar in the degrees of longitude: at the equator the degree of longitude measures 60 geographical miles; at Cairo, lat. 30° 6′, about 51 miles; at Petersburgh, lat. 60°, about 30 miles; and at the Magnetic Pole, lat. 70° 5′, only 20 miles:—thus gradually diminishing, until at the Pole itself, the circle having dwindled to a point, all magnitude has ceased.

§ 2. FOR GEOMETRICAL FIGURES *to exhibit the representative values of actual magnitude and space.*

When Lines have been measured and the angles taken which are formed by the lines drawn from any point or station to two or more objects in the field of view, we may enter the measurements and the angular magnitudes in a book: but besides this, we need some method of representing the relative positions and distances of object. Geometrical Drawing,—as in the Plans of the Architect and of the Engineer, or in the Maps of an Estate, a Parish, or a Kingdom,—is the means by which such a representation is effected. For the actual truth of a Demonstration in Geometry, it is of very little consequence whether we employ any figure at all, or whether the figure we do construct is perfectly accurate; but for the representation of distances and of forms it is of great importance that we should draw well all the lines and angles, in their due proportion, and in their proper position;—indeed, without this a map is a deception, and a plan given to a workman for his guidance, would only mislead and perplex him.

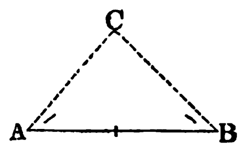
To ensure the accuracy so essential for all maps, plans, and working drawings, Scales of equal parts, and especially the Diagonal Scale, are constructed and employed: see pp. 51, 117. The principle acted on is a very simple one,—that in the same Map or Plan, whatever scale is made use of for any one line or distance, shall be adopted for all the lines and distances in that particular Map or Plan. Let the scale fixed on be an inch to represent a mile; then every inch of space on the plan or map, to which the scale is attached, will represent the value or distance of a mile: and again in a working drawing given to a mechanic for his guidance, let the scale agreed

on be half-an-inch to represent a foot, then every half-inch in the drawing will denote a foot in the thing to be constructed.

The Lines in a Map or Plan are representative only of the values of the real distances of objects;—but the Angles in a Map or Plan, if correctly drawn, are identical with the angles of the objects themselves; and the reason is, that the size of an angle does not depend on the length of the lines which form it, but on the narrowness or width of the opening between them. The instruments for transferring to a drawing the angles measured between any two objects and the angular point or station of observation, are principally the Circle or Protractor, and the Line of Chords; pp. 40 and 89.

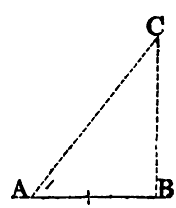
The *Methods for drawing nearly all Geometrical Figures*, are contained in the several Problems, pp. 188—207, which constitute the *first* and *second* of the leading divisions of the PRACTICAL RESULTS; but in the Application of them we must remember that lines and angles of a definite numerical value are given when we are required to construct Figures that shall show the positions, forms, and distances of objects. We subjoin therefore a few examples of the way in which such figures are to be drawn.

PROB. 1. *Given the two angles of a triangular field, equal respectively to 50° and 45°, and the interjacent side equal to 645 links; required by Construction the exact figure of the triangle, the amount of the other angle, and the value in links of the other two sides.*



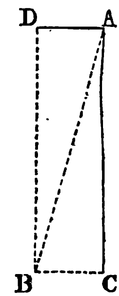
Take from the diagonal scale 645 equal parts, representative of 645 links, and of that length draw a line AB; at one extremity, A, make an angle of 50°; at the other extremity, B, an angle of 45°; produce the lines from A and B until they meet in C; ABC is the exact fig. required. If the angle at C be measured, it will be found to equal 85°; and from the same scale of equal parts, AC will be found to equal 458 links nearly, and BC 496 links nearly.

PROB. 2. *The distance* AB *from the foot of a precipice is* 288 *feet; at* A, *the angle made by a string* CA, *reaching to the top, with* AB, *is* 53° 8′; *required by construction the height of the precipice and the length of the string.*



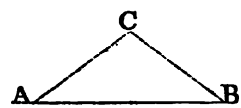
Draw a line AB containing 288 equal parts; at B erect a perpendicular, and at A draw an angle of 53° 8′; produce the lines AC, BC to meet in C:—AC, representing the length of the string, will measure 480 feet; and BC, the height of the precipice, 384 feet.

Prob. 3. *The breadth,* AD, *or* BC, *of the shaft of a well, of which the sides are parallel, is* 10 *feet; the angle of depression,* CAB, 15°: *required by construction,* AC, *the depth of the well.*



Take AD equal to 10 equal parts; and at A and D draw perpendiculars to AD; make the angle CAB 15°; and the point B, where AB and DB intersect, will be at the bottom of the shaft; through B draw BC parallel to DA; and CA equals DB; apply CA or DB to the same scale as AD, and the depth will be found 37·3 feet.

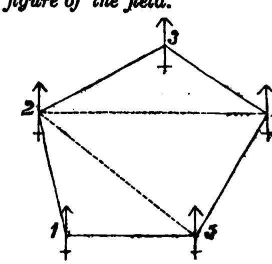
Prob. 4. *Two ships of war intending to cannonade a fort at* C, *separate from each other* 500 *yards; the angles between each ship and the fort and the other ship are,* A, 38° 16′, *and* B, 37° 9′: 300 *yards being a convenient distance for a cannonade, have the two ships taken up their stations at a proper distance?*



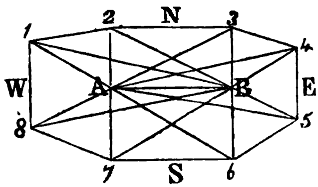
The line AB being drawn equal to 500, angle A to 38° 16′, and angle B to 37° 9′, and the triangle ACB completed;—it will be found, from the same scale, that AC measures 312, and BC 320 yards: the ships therefore are nearly at the exact distance required.

Prob. 5. *In surveying a field, the angles which the sides of the field made with the magnetic meridian, were observed at five stations, and the lengths of the sides measured: required to construct the figure of the field.*

Station	1.	North 17° West	262	links.
"	2.	North 62° East	324	"
"	3.	South 51° East	221	"
"	4.	South 28° West	296	"
"	5.	West	242	"



At Station 1 make an angle 17° west of North, and set off 262 equal parts; at Sta. 2, an angle 62° east of North, and set off 324; at Sta. 3, an angle 51° east of South, and set off 221; at Sta. 4, an angle 28° west of South, and set off 296; and at Sta. 5, a line due West, with a set off of 242; the figure will be completed: if now the diagonals and perpendiculars were drawn, the area of the field might be computed.



PROB. 6. For more extensive surveys two stations A and B are taken, the line AB is measured, and from a scale is set on the Plan; all the angles, which are formed by lines from the stations A and B to the objects at 1, 2, 3, 4, 5, 6, 7, and 8, are also measured and drawn; and the angular points being joined, an outline of the figure is obtained in the relative size or magnitude of its parts; and by taking any of the distances in the plan, and applying them to the same scale as that from which AB was set out, their equivalent, or representative values will also be obtained. Thus by drawing the diagonals, and dropping perpendiculars from the angular points which the diagonals subtend, the area of the figure may be computed from the scale. It must, however, be observed, that this method of computation is very liable to error, for it is difficult to construct an intricate figure so as to make it perfect in all its parts. Calculations from data obtained by actual measurements, are the only reliable methods of ascertaining with accuracy the areas of figures.

The foregoing examples will be sufficient to show how by the construction of figures we may exhibit the representative values of actual magnitudes and spaces. Other instances are given at pp. 51, 52, 54, 62, 83, 87, 90, 95, 96, 104, 107, 108, 117, 129, &c.

IV.—PRINCIPLES WHICH WITHOUT REQUIRING THAT WE SHOULD MEASURE ALL THE BOUNDARIES OF A SURFACE, YET ENABLE US ACCURATELY TO CALCULATE DISTANCES, MAGNITUDES, AND AREAS.

§ 1. *Lines, or Distances.*

1. The sides of an equilateral triangle, by Def. 24. I., of a square, by Def. 30. I., and of a rhombus, by Def. 32. I., are all equal; the measurement of one side in each, therefore, is sufficient for ascertaining the other sides.

2. In an isosceles triangle, by Def. 25. I., two sides are equal; therefore the measurement of one of those sides will give the other.

3. In parallelograms, by 34. I., the opposite sides are equal; it is necessary, therefore, only to measure any two conterminous sides.

4. Of the three sides, H hypotenuse, B base, and P perpendicular, any two being measured, the third may be calculated; for, by 47. I., $H=\sqrt{B^2+P^2}$; $B=\sqrt{H^2-P^2}$; and $P=\sqrt{H^2-B^2}$.

5. In a similar way, in a rectangle, of the two conterminous sides, AB, BC, and the diagonal, AC, any two being given, the third is found, by 47. I.; for $AC=\sqrt{AB^2+BC^2}$; $AB=\sqrt{AC^2-BC^2}$; and $BC=\sqrt{AC^2-AB^2}$.

6. If C, the circumference of a circle, or D, the diameter, be measured, the other may be found; for, by 41. I., Use 4, $C=D\times 3\cdot 1416$, & $D=\dfrac{C}{3\cdot 1416}$

7. When, of the three parts of a regular polygon, the side AB, the perpendicular on the side P, and the radius R, or distance from the centre to

the angular points, any two are measured, the other is found; for, 41 and 47. I., $\frac{AB}{2} = \sqrt{R^2 - P^2}$; $P = \sqrt{R^2 - \left(\frac{AB}{2}\right)^2}$; and $R = \sqrt{P +^2 \left(\frac{AB}{2}\right)^2}$

8. In any right-lined figure, if diagonals are drawn dividing it into triangles, and from the angular points perpendiculars are drawn to the diagonals,—on measuring the perpendiculars and the segments of each diagonal, then, by 47. I., the sides may be calculated.

9. In the case of *inaccessible distances*, the amount or calculation is obtained in various ways: as in 1. I., Use 4; 3. I., Use 3; 4. I., Use 2; 6. I., Use; 15. I., Use 1; 20. I., Use 2; 26. I., Use 1, 2, 3; 29. I., Use; 31. I., Use 2; 33. I.; 34. I., Use 4; 46. I., Use; 47. I., Use 4, 5; 6. II. Use.

§ 2. *Angles.*

1. In equilateral triangles, rectangles, squares, and regular polygons, the measurement of one angle is equivalent to the measurement of all; for, by Cor. 5. I., Def. 31, Def. 30, such figures have all their angles equal.

2. If we measure any two angles in a triangle, the third angle, by 32. I. will equal the supplement of 180°, or 180°—the sum of the two angles.

3. In a right-angled triangle, with one acute angle known, the other acute angle is equal to the complement of 90°, or 90°—the given acute angle.

4. When two st. lines intersect, by 15. I., the measure of any one angle gives the opposite vertical angle.

5. By measuring any two adjacent angles of a parallelogram, 34. I., we obtain the other two angles.

6. When a st. line cuts two parallel lines, the exterior angle equals the interior opposite angle, and the alternate angles are equal, 29. I; by measuring one angle, therefore, we know the other.

7. At the base of an isosceles triangle the angles are equal, 5. I.; one being measured, the other is known.

8. By approaching to, or receding from, a horizontal mirror, on which a given point from an object is reflected, we can ascertain the acute angle and the base of the right-angled triangle, 20. I., Use 2; and consequently we can calculate the vertical angle.

9. The parallax of a heavenly body is ascertained, 32. I., Use 1, by subtracting the zenith distance at the earth's centre from the zenith distance at the earth's surface.

10. When, by 4. I.; 8. I.; or 26. I.; two or more triangles are proved to have equal angles, the measurement or calculation of the angles of any one triangle is equivalent to the measurement or calculation of the angles of the others.

§ 3. *Magnitudes or Areas.*

1. In *squares*, the measurement of one side is sufficient,—for the altitude and base are equal; and the Area = the square of the side: 40. I. 3.

2. In all *parallelograms*, measure the base and altitude; for, by 35. I., Use, and 41. I. Use, the Area = the base × the altitude.

3. In all *triangles*, also measure the base and altitude, and take half the base; for, by 37. I., and 41. I., Use 1, the Area = ½ the base × the altitude.

4. All *right-lined figures* may be divided into triangles;—then the Area of the figure = the sum of the areas of the triangles: 41. I., Use 2.

5. Lines from the centre of a *regular polygon* divide it into equal triangles; its Area therefore = the Area of one triangle × the number of sides; or, = the perpendicular × $\frac{1}{2}$ the perimeter: 41. I., Use 3.

6. The Area of a *trapezium* is found, 40. I., Use 2, by taking half the sum of the parallel sides and the altitude, and multiplying the two quantities together.

7. In *circles*, measure the diameter D, and ascertain the circumference C; or measure the circumference, and ascertain the diameter; then, by 41. I., Use 4, the Area $= \frac{3 \cdot 1416 \times D^2}{4}$; or $= \frac{C^2}{4 \times 3 \cdot 1416}$; or $= \frac{D \times C}{4}$; or = the radius × the semi-circumference.

By the use of other Geometrical Truths, the Student might have a much more extended view of the Principles which assist Mathematical Calculations;—but many of those truths lie out of the limits of an Elementary Work; and enough has been advanced to show the wide Application and Utility of the First and Second Books of Euclid's Plane Geometry.

Indeed if any defence were required for confining the Examinations of Pupil Teachers, and of Scholars generally, in Elementary Schools, to the two books referred to, it is furnished by the very valuable Practical Results which have just been exhibited. Whoever has mastered and retains his familiarity with the Geometrical Principles now set before him, will possess sufficient knowledge of the subject for all the usual purposes of life,—and, what is more, will possess the means, whenever he chooses to employ them, of advancing with comparative facility to the higher and more abstruse mathematical learning. The right foundation has been laid; and the calls of professional duties and employments may be left to determine, whether the Student should remain satisfied with the mark attained, or go beyond it and labour in a wider field. If he is called, or prompted, to try the

more difficult paths, he will never regret that his attention in youth was chiefly confined to the Introductory Books of Plane Geometry. It is the accuracy and the thoroughness of the early training,—and not the wide extent of the subjects, traversed indeed, but not known,—which increase the power of the mind; and the true aim of the Teacher is to strengthen power by a smaller quantity well done, than to waste it on a multitude of projects, to none of which it is able to do justice. The steam, that sounds a thousand jerking whistles, does not perform half so much useful work as that which keeps in steady motion a single loom.

APPENDIX.

I.—Geometrical Analysis.

The Principles of Plane Geometry, as taught in Euclid's Elements, are established, by proceeding in a regular series from Definitions, Postulates, and Axioms already known, to the consequences which flow from, and which are dependent upon, the Definitions, Postulates, and Axioms. This Method is entirely one of building up, or of putting together, and is therefore named *Synthesis:* "it commences with what is given, and ends with what is sought,"—the materials being furnished, out of them it fashions a garment; it takes elementary substances, and forms a compound.

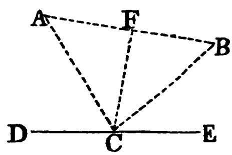
Analysis pursues an opposite course: it takes the compound, and resolves it into its constituent parts; the garment entire and completed is given for examination, and the aim is to discover of what it is made: Analysis begins with the thing sought, as a thing perfected and accomplished, and ends with whatever may have been supplied for the construction.

No aids except those derived from Geometry were admitted by the Ancients in conducting an Analysis; and therefore the term Geometrical Analysis is employed.

In analyzing a Problem, *the solution is assumed* to have been effected; and in analyzing a Theorem, *the truth* of the assertion contained in it is first of all admitted. When a Problem is analyzed, the object is—to discover something which, if done, would of necessity lead to the solution of the problem; and when a Theorem is analyzed, the object is—to determine whether the assertion is true or false.

An example will more clearly show the difference between the analytical and the synthetical Methods: for this purpose we take the Problem—

In a given st. line, DE, *to find a point,* C, *which shall be at the same distance from two other given points,* A *and* B.



By ANALYSIS.—We assume what was required, namely, that C, in DE, is the point equally distant from the given points A and B.

The line CA = the line CB; and AB being joined, the figure ACB is an isosc. △. If now the line AB is bisected in F, we have, in the two triangles, CA = CB, AF = BF, and FC common; ∴, by 8. I., △ AFC = △ BFC, and ∠ AFC = ∠ BFC. But when two points, as A and B, are given, the line joining them AB, is given; and the line being given, its middle point F, is also given; at F, that middle point, a perpendicular, FC, is given; and consequently, if produced, its point of intersection, C, with the given line DE. But C was the required point, and the production of the perpendicular from F determines it.

By SYNTHESIS.—Join the points A and B, and bisect AB in the point F; at F raise a perpendicular, and produce it to intersect DE in the point C;—that point C is equi-distant from A and B.

∵ AF = FB, FC common, and ∠ AFC = ∠ BFC.
∴ the side CA = the side CB: which was required.

In the Analysis, we have taken the Problem to pieces; in the Synthesis, we have put the parts together, and completed the purpose at which we aimed. It is by following similar Methods that other Geometrical Propositions may be analyzed and established. Analysis, however, is more suited for Problems;—Synthesis for Theorems.

The Rules for conducting an Analysis are few;—for the mode of procedure depends in a great measure on the knowledge and skill of the Student; and the greater these are, the greater facility and clearness will he manifest in making an analysis. It is a process which calls forth all the resources of his mind,—and therefore a very improving exercise for the young Geometrician.

The suggestions which are contained in RITCHIE's *Geometry*, p. 50, may be of service to the Learner, and therefore are recommended to his attention.

"1st. As in every other study, endeavour to ascertain what it is you have to do; examine into the nature and meaning of the Proposition, and form a clear, well-defined idea of the quantities concerned in the investigation.

2nd. Construct every figure with exactness, that the *eye* may aid the judgment.

3rd. It will often be necessary to join certain points so as to form equal triangles, isosceles or equilateral triangles, and other right-lined figures;

4th. Often, also, to lengthen lines, to draw perpendiculars from certain points in a line, or to let fall perpendiculars from certain points on straight lines.

5th. Often, straight lines or angles must be bisected, one angle joined to another, so as to get the sum of two angles; or a line drawn within an angle, so as to get the difference of two angles.

6th. Also, it will often be necessary to draw lines parallel to certain lines through remarkable points, which may be either given or required.

7th. In short, the Learner must form such a combination of lines, angles, and circles, as will, in his judgment, lead to the discovery of the object required. If, after trial, he finds he cannot reach the required point, and take the citadel by the path he has sketched out, he must commence the attack anew by following a different road, and by adopting a different system of *tactics.*"

"REMARK.—1. A point is said to be given, when its *position* is either given, or may be determined.

2. A line is given in *position* when its direction is given; in *magnitude*, when its length is given.

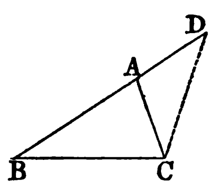
3. A line is given in *position and magnitude* when both its direction and magnitude are given.

4. The Position of a *point* can be found only—*first*, by a st. line cutting another st. line; *second*, by a st. line cutting the circumference of a circle; or, *third*, by the intersection of the arcs of two circles.

5. The position of a *line* is found, when any two points in it are found; and its Magnitude, when the extreme points are found."

A few EXAMPLES, selected from various sources, and restricted to the First and Second Books of the Elements, will show the Learner *how* an Analysis, or a Synthesis may be conducted.

Ex. 1.—PROB.—*Given an angle,* A; *a side,* BC, *opposite to it; and the sum,* BD, *of the other two sides of a triangle:—to construct the triangle.*



ANALYSIS.—The figure BAC is the required △, ∠ A the given angle, BC the side opposite to ∠ A, and BD the sum of the other two sides.

Join DC: the sides of the triangle BA and AC = the sum BD.

Take away the common part BA, and the rem. AC = the rem. AD; ∴ ∠ ACD = ∠ ADC.

But the ext. ∠ BAC = ∠ ACD + ∠ ADC, or ∠ BAC = 2 ∠ BDC; thus ∠ BDC = $\frac{1}{2}$ ∠ BAC.

Hence the method of construction.

SYNTHESIS.—At D, one extremity of BD, make an angle = $\frac{1}{2}$ ∠ BAC; and from B, the other extremity, draw BC, the given side, to meet DC in C; at C, make ∠ ACD = ∠ ADC, so that CA may meet BD in A: the triangle BAC will have its sides BA + AC = BD the sum; its ∠ = the given ∠, and its side = the given side.

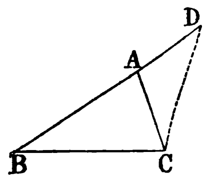
Ex. 2.—Prob.—*Given* BC *the base,* $\angle$ B *at the base, and* BD *the sum of the other two sides of a triangle:—to construct the triangle itself.*

Analysis.—The triangle ABC is the triangle required; for $\angle$ B equals the given angle, BC is the base, and BD = BA + AC, the sum of the sides.

On joining DC, ACD becomes an isosceles $\triangle$, and $\angle$ ACD = $\angle$ ADC.

Synthesis.—Make BC = the base, $\angle$ B = the given $\angle$ at the base, and BD = the sum of the sides.

Join CD, and at C make $\angle$ DCA = $\angle$ ADC; let CA meet BD in A; then the figure CAB is the triangle required.



Ex. 3.—Prob.—*From two given points,* A *and* B, *on the same side of a st. line given in position,* CD, *to draw two st. lines which shall meet in that line* CD, *and make equal angles with it.*

Analysis.—The point sought is G; and $\angle$ BGD = $\angle$ AGC.

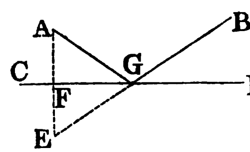
Produce BG until GE = GA, and join AE.

The $\angle$ EGC = $\angle$ BGD = $\angle$ AGC;

$\therefore$ $\angle$ EGF = $\angle$ AGF, and AG = GE, and FG is common, $\therefore$ $\angle$ AFG = $\angle$ EFG;

and they are rt. $\angle$s, and AF = FE.

But since A and CD are given, the perpendicular AF is given; and the point E is known, and G is determined by the intersection of CD and EB.



Synthesis.—From A draw a perpendicular to CD, and produce it, till FE = AF; join E, B, cutting CD in G; and draw AG: the lines AG and BG are the lines required.

For AF = FE, FG is common, and $\angle$ AFG = $\angle$ EFG;

$\therefore$ $\angle$ AGF = $\angle$ EGF = $\angle$ BGD; and $\therefore$ $\angle$ AGC = $\angle$ BGD.

Ex. 4.—Prob.—*To divide a given st. line,* AB, *into two such parts that the rectangle contained by them may be three-fourths of the greatest which the case will admit of.*

Analysis.—Let AB be the given line, and bisected in D; then $\square$ AD·DB, or the square on DB, is the greatest possible rectangle.

Assume C, in AB, as the point required, so that AC·CB = $\frac{3}{4}$ of DB^2.

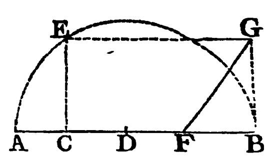
On AB describe a semicircle, and draw CE perpendicular from AB;

then the square on CE = $\square$ AC·CB.

Again, bisect DB in F; on FB describe the rt. $\angle$d. $\triangle$ in which FG = DB;

then BG is the line the square of which = $\frac{3}{4}$ of DB^2.

Hence, GB = EC; join G and E; $\therefore$ the points E and C are found.



SYNTHESIS.—Let AB be bisected in D, and DB in F; and on FB let the rt. ∠ d. △ FBG be made, having its hypotenuse FG = DB.

On AB describe a semicircle; through G draw GE ∥ AB, and through E, EC ∥ GB;

then C is the point required, the ▭ AC·CB being equal to $\frac{3}{4}$ of the sq. on DB, which is half the line AB.

Some Problems admit of only one definite solution, and they are called *determinate;* and some admit of two or more definite solutions, and these are called *indeterminate.*

A Problem is *determinate,* when it relates to the determination of *a single point,* and the data are sufficient to determine the position of that point: it is *indeterminate,* if, on omitting one or more of the conditions, the data which remain may be sufficient for the determination of *more than one point,* each of which satisfies the conditions of the problem. In general, such points are found to be situated in some line; and hence such line is called the *locus* of the point which satisfies the conditions of the problem.

Some Problems also are possible within certain limits; and certain magnitudes increase, while others decrease, within those limits; and after having reached a certain value, the former begin to decrease, while the latter increase. This circumstance gives rise to questions of *maxima* and *minima,* or the greatest and least values which certain magnitudes may admit of in indeterminate problems.—POTTS' *Euclid,* p. 289–292.

It may be remarked that Synthesis is well adapted for communicating a knowledge of the first principles of Geometry, or of any Science; but that Analysis is the great instrument or means of discovery,—it enables the Student to find out the solution of a problem for himself, and makes him more and more self-reliant in his search for Geometrical truth;—a most valuable power of mind, worth many a struggle to attain.

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II.—GEOMETRICAL EXERCISES.

The Skeleton Propositions, which form a continuation or completion of the Plan pursued in the Gradations of Euclid, furnish a most useful and improving series of Exercises; and, on the ascertained fact in the Art of Teaching, that Repetition is a most important auxiliary, they are recommended to the notice

and adoption of the Teachers of Mathematics. At any rate, where employed, they will accustom the student to be systematic and exact, and not to advance a step without a reason; and progress in this way, though it may chance to be less rapid, will be on a solid basis, and bring into play some of the most valuable qualities of the mind.

Potts, Colenso, Cooley, Chambers, and others, have each published Collections of Geometrical Exercises; but as Teachers may wish not to go beyond the limits of the present work, two Series of Exercises are now appended; the *First Series* consisting of Problems and Theorems which are inserted in the Gradations, and of which the General Enunciations may be given out to Learners for analysis, solution, or proof; and the *Second Series* containing Propositions which are not fully proved or not inserted in the Gradations.

Series I.

Problems in Book I.

1. By means of an equilateral triangle to measure an inaccessible distance.
2. To construct a scale of equal parts.
3. To ascertain an inaccessible distance when two sides and their included angle have been measured.
4. To show, by observations on the shadows which objects cast, the perpendicular heights of the objects.
5. To determine without a theodolite the angle at a given point made by a line from two objects meeting in that point.
6. To construct the Mariner's Compass-Card.
7. From a given point at the end of a st. line to raise a perpendicular.
8. On a given line to describe an isosceles triangle of which the perpendicular height is equal to the base.
9. From a given point over the end of a st. line to let fall a perpendicular to the line.
10. Given an angle of 73°, required its complement; an angle of 96°, required its supplement.
11. By the application of the principle that vertical angles are equal, to find the distance between two objects.
12. From a given point, A, to direct a ray of light against a mirror, so that the ray shall be reflected to another given point.
13. To determine the number and kind of polygons which may be joined so as to cover a given space.
14. To construct a triangle when the base, the less angle at the base, and the difference of the sides, are given.
15. By means of a mirror placed horizontally, to construct a triangle, the perpendicular of which shall be representative of the height of any object.

16. Given three lines respectively equal to 40, 50, and 30 equal parts, to form a triangle.
17. On a given line to describe an isosceles triangle having each of the equal sides double of the base.
18. On a given line and with a given side to construct an isosceles triangle.
19. To construct a Line or Scale of Chords.
20. By aid of a Line of Chords,—1°. to make an angle containing a certain number of degrees, as 40°; 2°. to measure a given angle; 3°. from the extremity of a line to raise a perpendicular; and 4°. to construct a triangle of which the base contains 40 equal parts, one of the angles at the base 40°, and its other adjacent side 35 equal parts; and to measure the other side and the other angles.
21. To measure an inaccessible distance, AB, only A being approachable.
22. By the theory of Representative Values to find the distance between two stations.
23. To measure an inaccessible distance, A B, neither A nor B being approachable.
24. Given the vertical angle and the perpendicular height of an isosceles triangle, to construct it.
25. On the principle that two rays of light proceeding from the centre of the sun to two points on the earth are physically parallel, to ascertain the earth's circumference.
26. By the method of parallel lines to ascertain the distance of an inaccessible object.
27. To determine the Parallax of a heavenly body.
28. To construct a figure which will give the representative value of the perpendicular height of a mountain.
29. To construct a regular Polygon,—1°. when the side is given; 2°. when the side is not given.
30. To ascertain both the perpendicular height of a mountain, and the horizontal distance from the foot of the mountain to the foot of the perpendicular.
31. To divide a finite st. line into any given number of equal parts.
32. To construct a Sliding Scale to measure the hundredth part of an inch.
33. To construct a Sliding Scale for measuring the minutes into which a degree on a circle is divided.
34. To ascertain the continuation of a st. line when an obstacle intervenes.
35. From a given point in the side of a parallelogram to bisect the parallelogram.
36. To convert a parallelogram into an equivalent rectangle.
37. By the method of Indivisibles to explain the equality of parallelograms on the same base and between the same parallels.
38. To construct a Diagonal Scale.
39. To divide a triangular space into two equal parts.
40. From any point in the side of a triangle to divide the triangle into two equal parts.
41. To find the Area of a trapezium.
42. To find the Area of a square.
43. To find the Area of a triangle.
44. To find the Area of any right-lined figure.
45. To find the Area of a regular polygon.

46. To find the Area of a circle.
47. Of the Diameter, Circumference, Area, and Ratio of the diameter and circumference, any two being given, to ascertain the others.
48. To describe a triangle equal to a given parallelogram, and having an angle equal to a given angle.
49. A parallelogram being given, to find another parallelogram equal to it, and having one side equal to a given line.
50. Given the area of one figure, and the side of another which is to be a parallelogram equal to the given figure, to find the other side of that parallelogram.
51. To change any right-lined figure, first into a triangle, and then into a rectangle of equal area.
52. To straighten a crooked boundary without changing the relative size of two fields.
53. Given the diagonal to construct a square.
54. To ascertain the height of an inaccessible object by aid of the Geometrical Square.
55. Given in numbers the sides of a right-angled triangle, to find the hypotenuse.
56. Given in numbers the hypotenuse and one side, to find the other side.
57. To find a square equal to any number of squares; or a square that is the multiple of a given square; or a square that equals the difference of two squares; or a square that is the half, the fourth, &c., of a given square.
58. To make a rectilineal figure similar to a given rectilineal figure.
59. To make a circle the double, or the half, of another circle.
60. To construct the Chords, Natural Sines, Tangents, and Secants, of Trigonometrical Tables.
61. To find right triangular numbers.
62. To compute Heights and Distances from the curvature of the earth.

Theorems in Book I.

1. By the bisection of the vertical angle of an isosceles triangle to show that the angles at the base are equal; and also that the bisecting line bisects the base at right angles.
2. Only one perpendicular can be drawn from a given point to a given st. line.
3. The perpendicular is the shortest line from a given point to a given st. line.
4. From the same point only two equal lines can be drawn to a given st. line.
5. All heavy bodies free to move continually descend, or seek the point which is nearest to the earth's centre.
6. Of all lines that can be drawn from one point to a plane surface, and reflected to a third point, those are the shortest which make the angle of incidence equal to the angle of reflection.
7. The chord of 60° is equal to the radius of the circle.

8. A line which is perpendicular to one parallel, is also perpendicular to the other.
9. If a line falling on two other lines make the interior angles on the same side less than two rt. angles, those two lines on being produced shall intersect.
10. A parallel to the base of a triangle through the point of bisection of one side, will bisect the other side.
11. The lines which join the middle points of the three sides of a triangle, divide it into four triangles which are equal in every respect.
12. The line joining the points of bisection of each pair of sides of a triangle, is equal to half the third side.
13. A trapezium is equal in area to a parallelogram of the same altitude, and of which the base is half the sum of the parallel sides.
14. The squares on equal lines are equal; and if the squares are equal, the lines are equal.
15. Every parallelogram having one rt. angle, has all its angles rt. angles.
16. If a perpendicular be drawn from the vertex of a triangle to the base, the difference of the squares of the sides is equal to the difference between the squares of the segments.
17. If a perpendicular be drawn from the vertex of a triangle to the base, or to the base produced, the sums of the squares of the sides and of the alternate angles are equal.

Problems in Book II.

1. From Propositions 1, 2, and 3, deduce various methods for the Multiplication of Numbers, and demonstrate the rule.
2. From Prop. 4, point out a practical way of extracting the Square root of a number, and prove the correctness of the formula.
3. To find the difference between the squares of two unequal numbers without squaring them.
4. To find Quantities in Arithmetical Progression.
5. To find the value of an Adfected Quadratic Equation in Algebra.
6. By aid of Prop. 6, to ascertain the diameter of the earth.
7. Given the sum and the difference of two magnitudes, to find the magnitudes themselves.
8. From the Area of a rectangle and one side given, to obtain the other side.
9. To divide a given Line a, so that its parts x and $a—x$ may make $a(a—x) = x^2$. Let the solution be given both algebraically and arithmetically.
10. To ascertain the Area of a triangle when the three sides are known.
11. From the three sides of a triangle given, to obtain the perpendicular;— 1°. when the perp. falls within the base; and 2°. when it falls without the base.
12. To find a mean proportional to two given lines.
13. To approximate to the square of any curve-lined figure.
14. To calculate the Area of any right-lined figure.

Theorems in Book II.

1. The difference of the squares of two quantities, equals the rectangle of their sum and difference.
2. The difference of the squares of two quantities is greater than the square of their differenee, by twice the rectangle of the less and their difference.
3. The square of the sum of two lines is equal to four times the rectangle under them, together with the square of their difference.
4. Four times the square of half the sum is equal to four times the rectangle under the lines, together with four times the square of half the difference.
5. The sum of the squares of any two lines is equal to twice the square of half their sum, together with twice the square of half their difference.
6. The sum of the squares is equal to half the square of the sum, together with half the square of the difference.

SERIES II.

PROPOSITIONS NOT FULLY PROVED, OR NOT INSERTED IN THE GRADATIONS.

Problems.—Book I.

1. To find a point which is equidistant from the three vertical points of a triangle.
2. To bisect a triangle by a line drawn from a given point in one of its sides.
3. Describe a circle which shall pass through two given points, and have its centre in a given line.
4. Through a given point to draw a line that shall be equally inclined to two given lines.
5. Given a triangle ABC, and a point D in AB; to construct another triangle ADE equal to the former, and having the common angle A.
6. To change a triangle into another equal triangle of a given altitude.
7. To draw a line which, if produced, would bisect the angle between two given lines, without producing them to meet.
8. To trisect a right angle.
9. To trisect a given st. line.
10. Given the sum of the sides of a triangle, and the angles at the base, to construct it.
11. Given the diagonal of a square, to construct the square of which it is the diagonal.

12. Given the sum and difference of the hypotenuse and a side of a right-angled triangle, and also the remaining side, to construct it.
13. To find the *locus* of all points which are equidistant from two given points.

Theorems.—Book I.

1. In an isosceles triangle, the right line which bisects the vertical angle also bisects the base, and is perpendicular to the base.
2. If four lines meet at a point, and make the opposite vertical angles equal, each alternate pair of lines will be in the same st. line.
3. The difference of any two sides of a triangle is less than the remaining side.
4. Each angle of an equilateral triangle is equal to one-third of two right angles, or to two-thirds of one right angle.
5. The vertical angle of a triangle is right, acute, or obtuse, according as the line from the vertex bisecting the base is equal to, greater, or less than half the base.
6. If the opposite sides or opposite angles of a quadrilateral be equal, the figure is a parallelogram.
7. If the four sides of a quadrilateral are bisected, and the middle points of each pair of conterminous sides joined by st. lines, those joining lines will form a parallelogram the area of which is equal to half that of the given quadrilateral.
8. If two opposite sides of a parallelogram be bisected, and two lines be drawn from the points of bisection to the opposite angles, these two lines trisect the diagonal.
9. In any right-angled triangle, the middle point of the hypotenuse is equally distant from the three angles.
10. The square of a line is equal to four times the square of its half.
11. The st. line which bisects two sides of a triangle, is parallel to the third side, and equal to one-half of it.
12. If two sides of a triangle be given, its area will be greatest when they contain a rt. angle.
13. Of equal parallelograms that which has the least perimeter is the square.
14. The area of any two parallelograms described on the two sides of a triangle, is equal to that of a parallelogram on the base, whose side is equal and parallel to the line drawn from the vertex of the triangle to the intersection of the two sides of the former parallelograms produced to meet.
15. The vertical angle of a triangle is acute, rt. angled, or obtuse, according as the square of the base is less than, equal to, or greater than, the sum of the squares of the sides.

Problems.—Book II.

1. The sum and difference of two magnitudes being given, to find the magnitudes themselves.
2. To describe a square equal to the difference of two given squares.
3. To divide a given line into two parts, such that the squares of the whole line and of one of the parts shall be equal to twice the square of the other part.
4. To divide a given line into two such parts that the rectangle contained by them may be three-fourths of the greatest of which the case admits.
5. Given the area of a right-angled triangle, and its altitude or perpendicular from the vertex of the rt. angle to the opposite side, to find the sides.
6. Given the segments of the hypotenuse made by the perp. from the rt. angle, to find the sides.
7. To divide a line internally, so that the rectangle under its segments shall be of a given magnitude.
8. To cut a line externally, so that the rectangle under the segments shall be equal to a given magnitude, as the square on A.
9. Given the difference of the squares of two lines and the rectangle under them, to find the lines.
10. There are five quantities depending on a rectangle,—1°. the sum of the sides; 2°. the difference of the sides; 3°. the area; 4°. the sum of the squares of the sides; and 5°. the difference of the squares of the sides:—by combining any two of these five quantities, find the sides of the rectangle.

Theorems.—Book II.

1. The square of the perpendicular upon the hypotenuse of a right-angled triangle drawn from the opposite angle, is equal to the rectangle under the segments of the hypotenuse.
2. The squares of the sum and of the difference of two lines, are together double of the squares of these lines.
3. In any triangle the squares of the two sides are together double of the squares of half the base, and of the line joining its middle point with the opposite angle.
4. The square of the excess of one st. line above another, is less than the squares of the two st. lines by twice their rectangle.
5. The squares of the diagonals of a parallelogram are together equal to the squares of the four sides.
6. If a st. line be divided into two equal and also into two unequal parts, the squares of the two unequal parts are together equal to twice the rectangle contained by these parts, together with four times the square of the line between the points of section.
7. If a st. line be drawn from the vertex of a triangle to the middle point of the opposite side, the sum of the squares of the other sides is equal

to twice the sum of the squares of the bisector and half of the bisected side.

8. The sum of the squares of the sides of a quadrilateral figure is equal to the sum of the squares of the diagonals, together with four times the square of the line joining their points of bisection.

9. If st. lines be drawn from each angle of a triangle bisecting the opposite side, four times the sum of the squares of these lines is equal to three times the sum of the squares of the side of the triangle.

10. The square of either of the sides of the rt. angle of a rt. angled triangle, is equal to the rectangle contained by the sum and difference of the hypotenuse and the other side.

11. If from the middle point C, of a st. line AB, a circle be described, the sums of the squares of the distances of all points in this circle from the ends of the st. line AB, are the same; and those sums are equal to twice the sum of the squares of the radius and of half the given line.

12. Prove that the sum of the squares of two lines is never less than twice their rectangle; and that the difference of their squares is equal to the rectangle of their sum and difference.

13. If, within or without a rectangle, a point be assumed, the sum of the squares of lines drawn from it to two opposite angles, is equal to the sum of the squares of the lines drawn to the other two opposite angles.

14. If the sides of a triangle be as 4, 8, and 10, the angle which the side 10 subtends will be obtuse.

15. If in a rt. angled triangle a perpendicular be drawn from the rt. angle to the hypotenuse, the rectangle of one side and of the non-adjacent segment of the hypotenuse, shall equal the rectangle of the other side and of the other non-adjacent segment of the hypotenuse.

N.B.—A Key to the Geometrical Exercises of the Appendix will be published immediately. The Appendix will also be sold separately, to be used as a Book of Exercises.

FINIS.